GEOMETRIC CONDITIONS FOR THE RECONSTRUCTION OF A HOLOMORPHIC FUNCTION BY INTERPOLATION FORMULA

AMADEO IRIGOYEN

ABSTRACT. We give here some precisions and improvements about the validity of the explicit reconstruction of any holomorphic function on $\mathbb{C}^2$ from the knowledge of its restrictions on a family of complex lines. Such a validity depends on the mutual repartition of the lines. This condition can by geometrically described and is equivalent to a stronger stability of the reconstruction formula in terms of permutations and subfamilies of lines. The motivation of this problem also comes from possible applications in mathematical economics and medical imaging.

CONTENTS

1. INTRODUCTION

In this paper we give some results about the validity of the explicit reconstruction of any holomorphic function on $\mathbb{C}^2$ from its restrictions on a family of complex lines that cross the origin. Such a family can be written as

$$\left\{z \in \mathbb{C}^2,\, z_1 - \eta_j z_2 = 0\right\}_{j\geq 1} ,$$

where the directions $\eta_j \in \mathbb{C}$ are all differents (we forget the special line $\{z_2 = 0\}$). We remind from [11] and [12] the following interpolation formula:

$$(1.1)\quad E_N(f;\eta)(z) := \sum_{p=1}^{N}\left(\prod_{j=p+1}^{N}(z_1-\eta_j z_2)\right)\sum_{q=p}^{N}\frac{1+\eta_p\overline{\eta_q}}{1+|\eta_q|^2}\frac{1}{\prod_{j=p,j\neq q}^{N}(\eta_q-\eta_j)}\times$$

$$\times\sum_{m\geq N-p}\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{m-N+p}\frac{1}{m!}\frac{\partial^m}{\partial v^m}|_{v=0}[f(\eta_q v, v)]\,,$$

where $N \geq 1$ and $z = (z_1, z_2) \in \mathbb{C}^2$. For all $N \geq 1$ and $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, $E_N(f;\eta)$ satisfies the following properties:

Supported by projects MTM2008-05561-C02-01 and 2005SGR00611.

- $E_N(f;\eta) \in \mathcal{O}\left(\mathbb{C}^2\right)$;
- $E_N(f;\eta)$ is an explicit formula that is constructed with the data
$$\left\{f_{|\{z_1=\eta_j z_2\}}\right\}_{1\leq j\leq N} \ ;$$
- $\forall\, j=1,\ldots,N$, $E_N(f;\eta)_{|\{z_1=\eta_j z_2\}} = f_{|\{z_1=\eta_j z_2\}}$;
- $\forall\, P \in \mathbb{C}[z_1,z_2]$ with $\deg P \leq N-1$, $E_N(P;\eta) \equiv P$.

The essential problem is that there is generally no guarantee that, as $N\to\infty$, $E_N(f;\eta)$ will converge to f (although it coincides with f on an increasing number of lines). We know from [12] that it is false, ie there are families of lines with (at least) an associate function such that $E_N(f;\eta)$ will not converge. Since we are interested in a reconstruction formula whose convergence is guaranted for every function $f\in\mathcal{O}\left(\mathbb{C}^2\right)$, we want to determine all the *good* families of lines for which the convergence of the associate interpolation formula is made sure for any function. In this case, we just say the interpolation formula $E_N(\cdot;\eta)$ converges, ie $E_N(f;\eta)$ converges for all $f\in\mathcal{O}\left(\mathbb{C}^2\right)$. At the contrary, we will say that $E_N(\cdot;\eta)$ does not converge if there is $f\in\mathcal{O}\left(\mathbb{C}^2\right)$ such that $E_N(f;\eta)$ does not converge. This yields to the following result from [12]:

Theorem 1. *Let $\{\eta_j\}_{j\geq1}$ be bounded. Then the interpolation formula $E_N(f;\eta)$ converges to f, uniformly on any compact $K\subset\mathbb{C}^2$ and for all $f\in\mathcal{O}\left(\mathbb{C}^2\right)$, if and only if there exists R_η (that only depends on the sequence $\{\eta_j\}_{j\geq1}$) such that, for all p, $q\geq0$, one has*

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right| \quad\leq\quad {R_\eta}^{p+q}\,. \tag{1.2}$$

The operator Δ_p is defined as follows: for any function h that is just defined on the points η_j, $j\geq1$, one has

$$\begin{cases}\Delta_0(h)(\eta_1)=h(\eta_1)\,;\\ \forall\, p\geq1,\ \Delta_{p,(\eta_p,\ldots,\eta_1)}(h)(\eta_{p+1}) = \dfrac{\Delta_{p-1,(\eta_{p-1},\ldots,\eta_1)}(h)(\eta_{p+1})-\Delta_{p-1,(\eta_{p-1},\ldots,\eta_1)}(h)(\eta_p)}{\eta_{p+1}-\eta_p}\end{cases}$$

($\Delta_p(h)$ means the discrete derivative of order p of the function h).

Theorem 1 can also be reformulated in the more general case when the set $\{\eta_j\}_{j\geq1}$ is not dense.

The difficulty to understand the operator Δ_p that is constructed by induction (in particular if we want to compute it numerically), gives us the motivation to find a criterion that could be more natural to understand. This yields to the following definition:

Definition 1. *The set $\{\eta_j\}_{j\geq1}$ is* locally interpolable by real-analytic curves *if, for all $\zeta\in\overline{\{\eta_j\}_{j\geq1}}$ (topological closure of $\{\eta_j\}_{j\geq1}$), there exist a neighborhood V of ζ and a smooth real-analytic curve $\mathcal{C}$ such that $\zeta\in\mathcal{C}$ and*

$$V\bigcap\{\eta_j\}_{j\geq1} \quad\subset\quad \mathcal{C}\,.$$

Equivalently, for all $\zeta\in\overline{\{\eta_j\}_{j\geq1}}$, there exist a neighborhood V of ζ and $g\in\mathcal{O}(V)$ such that, $\forall\,\eta_j\in V$,

$$\overline{\eta_j} \quad=\quad g\left(\eta_j\right)\,. \tag{1.3}$$

One has the following result:

Theorem 2. *If $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves, then the interpolation formula $E_N(f,\eta)$ converges to f uniformly on any compact $K\subset\mathbb{C}$ and for all $f\in\mathcal{O}\left(\mathbb{C}^2\right)$.*

This new geometric condition is more natural to understand and to be formulated. Nevertheless, it is a sufficient criterion that is not equivalent. Indeed, there are families of lines that are not locally interpolable by real-analytic curves and whose associate formula $E_N(\cdot;\eta)$ converges for every $f\in\mathcal{O}\left(\mathbb{C}^2\right)$. Proposition 3 from Section 2 gives an example of such a family: it consists on constructing a sequence $(\eta_j)_{j\geq 1}$ whose topological closure is the whole square $[0,1]+i[0,1]$, but in such a way that, for all $N\geq 1$, the first N points η_j are the less close possible from each other (this construction intrinsically uses the ε-entropy of the 2-dimensional square, cf [15]).

Finally, there is a special case of equivalence: if $\eta=(\eta_j)_{j\geq 1}$ is a convergent sequence, then its associate interpolation formula $E_N(\cdot;\eta)$ converges if and only if it is locally interpolable by real-analytic curves (Section 2, Proposition 2). As applied, this gives us an easy way to construct such families of complex lines whose associate interpolation formula does not converge (Section 2, Corollary 2).

This last conclusion yields to the following question: why is this geometric criterion not (always) necessary? What is the difference between the case of the square and the one of a convergent sequence? The answer yields to another question: what is the action of the group of the permutations $\mathfrak{S}_{\mathbb{N}}$ on the validity of the convergence of $E_N(\cdot;\eta)$?

We remind that $E_N(f;\eta)$ is constructed with the first N lines associate to $\eta_1,\ldots,\eta_N$ (see (1.1)). The modified formula $E_N(f;\sigma(\eta))$ is the same one that is constructed with the first N lines $\eta_{\sigma(1)},\ldots,\eta_{\sigma(N)}$. Since any permutation does not change the family $\{\eta_j\}_{j\geq 1}$, one is tempted to think that $E_N(f;\sigma(\eta))$ and $E_N(f;\eta)$ are essentially the same. For example, if $M_N:=\max\{N,\sigma(1),\ldots,\sigma(N)\}$, then $E_{M_N}(f;\eta)$ and $E_{M_N}(f;\sigma(\eta))$ both interpolate f on the M_N lines $\eta_1,\ldots,\eta_N$ and $\eta_{\sigma(1)},\ldots,\eta_{\sigma(N)}$. Then we could believe that the action of the permutation group does not change the validity of the convergence of the associate formula $E_N(\cdot;\eta)$. It is obvious if σ only changes a finite number of lines but unfortunately it is not the case for all $\sigma\in\mathfrak{S}_{\mathbb{N}}$.

Any permutation of a convergent sequence will still be a convergent sequence (with the same limit). Analogously, any permutation of a sequence that is locally interpolable by real-analytic curves will not change this property (since this is a set-condition that does not depend on any numeration). On the contrary, if we completely change the order of the sequence of the square from Proposition 3, the associate interpolation formula may not converge any more. The following result from Section 3, Subsection 3.1 shows that the permutations can essentially change everything about the validity of the convergence of the associate interpolation formula $E_N(\cdot;\eta)$.

Proposition 1. *Consider $(\theta_j)_{j\geq 1}$ and $(\kappa_j)_{j\geq 1}$ two sequences such that:*

- *$(\theta_j)_{j\geq 1}$ (resp. $(\kappa_j)_{j\geq 1}$) is a bounded sequence whose interpolation formula $E_N(\cdot;\theta)$ (resp. $E_N(\cdot;\kappa)$) does not converge (resp. converges);*
- *dist* $\left(\{\theta_j\}_{j\geq 1},\{\kappa_j\}_{j\geq 1}\right)>0$.

Now consider $(\eta_j)_{j\geq 1}$ *the sequence defined as follows:*

$$\eta_j \quad := \quad \begin{cases} \theta_{j/2} \text{ if } j \text{ is even,} \\ \kappa_{(j+1)/2} \text{ if } j \text{ is odd.} \end{cases}$$

Then there exist σ_1 *and* $\sigma_2 \in \mathfrak{S}_{\mathbb{N}}$ *such that* $E_N(\cdot;\sigma_1(\eta))$ *converges (ie for all* $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, $E_N(f;\sigma_1(\eta))$ *converges to* f *uniformly on any compact* $K \subset \mathbb{C}^2$*) and* $E_N(\cdot;\sigma_2(\eta))$ *does not (ie there is - at least - one function* $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ *such that* $E_N(f;\sigma_2(\eta))$ *does not converge uniformly on any compact* $K \subset \mathbb{C}^2$*).*

Such families exist and can be easily given: one can consider for example the sequence $\eta = (\eta_j)_{j\geq 1}$ constructed with

$$\theta_j \;=\; \frac{i^j}{j} \quad \text{and} \quad \kappa_j \;=\; 3 + \sin{(j)} \tag{1.4}$$

(Section 2, Corollary 3). This result also gives an effective way to get counterexamples for the reciprocal sense of Theorem 2 (Section 2, Corollary 4): there are sequences $\eta = (\eta_j)_{j\geq 1}$ that are not locally interpolable by real-analytic curves and whose associate interpolation formula converges.

Then we know that the action of the permutations can have a strong influence for the convergence of the associate interpolation formula. On the other hand, special sets like convergent sequences (resp. sets that are locally interpolable by real-analytic curves) stay inert under this action. This yields to the following question: a given family $\{\eta_j\}_{j\geq 1}$ whose associate interpolation formula $E_N(\cdot;\sigma(\eta))$ always converges under the action of any permutation σ, should be locally interpolable by real-analytical curves? The answer is affirmative, as specified by the following result whose proof is given in Section 3, Subsection 3.4.

Theorem 3. *The family* $\eta = \{\eta_j\}_{j\geq 1}$ *is locally interpolable by real-analytic curves if and only if, for all* $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ *and all* $\sigma \in \mathfrak{S}_{\mathbb{N}}$, $E_N(f;\sigma(\eta))$ *converges to* f *uniformly on any compact subset* $K \subset \mathbb{C}^2$.

First, this result finally gives an equivalent condition in terms of the validity of the convergence of this interpolation formula. In particular, one can see why this geometric condition is sufficient but also too strong in general.

Next, this is an equivalence between a geometric property of a given set and the validity of the convergence of its associate interpolation formula (ie in terms of functional approximation theory).

Lastly, in its proof we use an intermediate result about the stability of $E_N(\cdot;\eta)$ by subsequences (Section 3, Proposition 4). This result claims that, if a sequence $\eta = (\eta_j)_{j\geq 1}$ makes converge its associate interpolation formula and we take away a bounded subsequence whose density is at most fractional (ie, for all $N \geq 1$, the number η_{j_k} from the subsequence with $j_k \leq N$, is at most N/C where $C > 1$), then the remaining sequence still makes converge its associate interpolation formula. Notice that the condition that the subsequence is bounded is used in order to rigorously prove the proposition, but it may still be true in a more general case.

Proposition 4 also comes from the motivation of the following question: if the formula $E_N(\cdot;\eta)$ is convergent, what about $E_N(\cdot;\eta')$, where η' is any given (infinite) subsequence of η? A first answer would be affirmative because of the following intuitive argument: if $E_N(f;\eta)$ can interpolate f in more lines than $E_{N'}(f;\eta')$ does

(where $\{\eta_{j_1}, \ldots, \eta_{j_{N'}}\} = \{\eta_1, \ldots, \eta_N\} \cap \eta')$ and moreover $E_N(f;\eta)$ converges, then why should not $E_N(f;\eta')$ too? Unfortunately this is not the case (Section 3, Corollary 5). One can give simple examples like the following one that is an application of Proposition 1: if we consider the above sequence from (1.4) with its associate permutation σ_1 (ie such that $E_N(\cdot;\sigma_1(\eta))$ converges), then the subsequence $(\eta_{\sigma_1(j_k)})_{k\geq 1}$ that will exactly get back the sequence $(\theta_k)_{k\geq 1}$, will make diverge its associate interpolation formula.

Conversely, if consider the same sequence but this time with its associate permutation σ_2 (so that its associate interpolation formula does not converge), then the subsequence that will get back the sequence $(\kappa_j)_{j\geq 1}$ will make converge its associate interpolation formula.

It follows that, as in the case of the permutation group, the action of taking subsequences can change everything, in one sense as in the other one. In fact, this stability by subsequences is a strong condition too, as specified by the following result that claims that it is an equivalent condition to the geometric criterion (1.3).

Theorem 4. *The sequence $\eta = (\eta_j)_{j\geq 1}$ is locally interpolable by real-analytic curves if and only if, for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ and all subsequence $(\eta_{j_k})_{k\geq 1}$, the associate interpolation formula $E_N\left(\cdot\,;(\eta_{j_k})_{k\geq 1}\right)$ converges to f (uniformly on any compact subset).*

We finish this paper by giving some affirmative results for the case where the family η is dense. We remind that, in order to apply Theorem 1, we need to assume that the family η is not dense. We already know that a dense set cannot be locally interpolable by real-analytic curves since its topological closure has nonempty interior (Section 2, Lemma 1). It follows that, a dense sequence η being given, one has one of two things: either the associate interpolation formula $E_N(\cdot;\eta)$ does not converge; either it converges but then by Theorem 3 there is a permutation $\sigma \in \mathfrak{S}_{\mathbb{N}}$ such that $E_N(\cdot;\sigma(\eta))$ does not converge. Nevertheless this does not tell us if there ever exists a dense sequence whose associate interpolation formula converges. The answer is affirmative and is a consequence of the following result that is proved in Section 4.

Theorem 5. *Let $\eta = (\eta_j)_{j\geq 1}$ be any dense sequence. Then there exists $\sigma_c \in \mathfrak{S}_{\mathbb{N}}$ (resp. $\sigma_d \in \mathfrak{S}_{\mathbb{N}}$) such that the interpolation formula $E_N(\cdot;\sigma_c(\eta))$ converges (resp. $E_N(\cdot;\sigma_d(\eta))$ does not).*

In particular, one can deduce as an application the following result.

Corollary 1. *There exist dense sequences $\eta = (\eta_j)_{j\geq 1}$ whose associate interpolation formula $E_N(\cdot;\eta)$ converges (ie for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, $E_N(f;\eta)$ converges to f, uniformly on any compact subset).*

To finish this paper, we still do not have an equivalent analytic criterion for the case of a dense sequence. We just know that condition (1.2) from Theorem 1 is always necessary.

We think that an equivalent analytic criterion for a dense sequence should be the following: the interpolation formula $E_N(\cdot;\eta)$ converges if and only if $(\eta_j)_{j\geq 1}$

and $(1/\eta_j)_{j\geq 1}$ both satisfy condition (1.2) from Theorem 1 (see Remark 4.1 from Section 4).

I would like to thank G. Henkin for having introduced me this interesting problem and J. Ortega-Cerda for all his guidance, rewarding ideas and discussions to progress in.

2. The non-equivalence of the geometric criterion

2.1. **A special case of equivalence.** We begin with the following lemma that will be usefull in the next subsection and Section 4.

Lemma 1. *The topological closure of a set that is locally interpolable by real-analytic curves, has empty interior.*

Proof. Assume that it is not the case. Then there are $\zeta_0 \in \mathbb{C}$ and $\varepsilon_0 > 0$ such that $D(\zeta_0, \varepsilon_0) \subset \overline{\{\eta_j\}_{j\geq 1}}$. In particular, ζ_0 cannot be isolated. Since $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves, there are $V_{\zeta_0} \in \mathcal{V}(\zeta_0)$ and $g_{\zeta_0} \in \mathcal{O}(V_{\zeta_0})$ such that, for all $\eta_j \in V$, $g_{\zeta_0}(\eta_j) = \overline{\eta_j}$. By reducing V_{ζ_0} if necessary, one can assume that V_{ζ_0} is open and $V_{\zeta_0} \subset D(\zeta_0, \varepsilon_0)$. In particular, for any subsequence $(\eta_{j_k})_{k\geq 1}$ that converges to ζ_0 with $\eta_{j_k} \neq \zeta_0$, $\forall\, k \geq 1$ (such subsequences exist since ζ_0 is not isolated), one has for all k large enough (so that $\eta_{j_k} \in V_{\zeta_0}$)

$$\frac{\overline{\eta_{j_k} - \zeta_0}}{\eta_{j_k} - \zeta_0} = \frac{\overline{\eta_{j_k}} - \overline{\zeta_0}}{\eta_{j_k} - \zeta_0} = \frac{g(\eta_{j_k}) - g(\zeta_0)}{\eta_{j_k} - \zeta_0} \xrightarrow[k\to\infty]{} \frac{\partial g}{\partial \zeta}(\zeta_0)\,.$$

In particular $\left|\dfrac{\partial g}{\partial \zeta}(\zeta_0)\right| = 1$ then one has $\dfrac{\partial g}{\partial \zeta}(\zeta_0) = e^{i\theta}$, $\theta \in \mathbb{R}$. We set

$$w_p = \zeta_0 + \frac{1}{p}\, i\, e^{-i\theta/2}$$

with $p \geq p_0$ and p_0 large enough such that $w_p \in V_{\zeta_0}$. Since $\{w_p\}_{p\geq p_0} \subset V_{\zeta_0} \subset \overline{\{\eta_j\}_{j\geq 1}}$, for all $p \geq p_0$, there is $\eta_{j_p} \in \{\eta_j\}_{j\geq 1}$ such that $\eta_{j_p} \in V_{\zeta_0}$ and

$$\left|\eta_{j_p} - w_p\right| \leq \frac{1}{2p^2}\,.$$

Then (since in particular $(\eta_{j_p})_{p\geq p_0}$ converges to ζ_0)

$$\begin{aligned} e^{i\theta} = \frac{\partial g}{\partial \zeta}(\zeta_0) &= \lim_{p\to\infty} \frac{\overline{\eta_{j_p}} - \overline{\zeta_0}}{\eta_{j_p} - \zeta_0} = \lim_{p\to\infty} \frac{\overline{w_p} - \overline{\zeta_0} + \overline{\eta_{j_p}} - \overline{w_p}}{w_p - \zeta_0 + \eta_{j_p} - w_p} \\ &= \lim_{p\to\infty} \frac{-ie^{i\theta/2}/p + O(1/p^2)}{ie^{-i\theta/2}/p + O(1/p^2)} = -e^{i\theta}\,, \end{aligned}$$

and that is impossible.

√

Now we prove the following result that claims that, in the particular case when $(\eta_j)_{j\geq 1}$ is convergent sequence, the geometric condition (1.3) becomes necessary.

Proposition 2. *Assume that $(\eta_j)_{j\geq 1}$ is a convergent sequence and let be $\eta_\infty = \lim_{j\to\infty} \eta_j$. If the interpolation formula $E_N(\cdot;\eta)$ converges, then $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves.*

Proof. First, the set $\{\eta_j\}_{j\geq 1}$ is bounded for being a convergent sequence in $\mathbb{C}$. On the other hand, one has

$$\overline{\{\eta_j\}_{j\geq 1}} \quad = \quad \{\eta_\infty\} \cup \{\eta_j\}_{j\geq 1} \, .$$

It follows that, for all $\zeta_0 \in \overline{\{\eta_j\}_{j\geq 1}}$, either $\zeta_0 = \eta_\infty$, either $\exists\, j_0 \geq 1$ such that $\zeta_0 = \eta_{j_0} \neq \eta_\infty$. In the last case, η_{j_0} is isolated and the assertion is obvious by choosing $V_{\eta_{j_0}}$ such that $V_{\eta_{j_0}} \cap \{\eta_j\}_{j\geq 1} = \{\eta_{j_0}\}$ and $g(\zeta) = \overline{\eta_{j_0}}$. So one can assume in the following that $\zeta_0 = \eta_\infty$. Our goal is to prove that $\{\eta_j\}_{j\geq 1}$ satisfies condition (1.3) in a neighborhood of η_∞.

Since $E_N(f;\eta)$ converges on any compact subset and for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, it follows by Theorem 1 that there is R_η such that, $\forall\, p,\, q \geq 0$, one has

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right| \quad \leq \quad {R_\eta}^{p+q} \, .$$

In particular, one has for all $p \geq 0$

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\zeta \mapsto \varphi(\zeta)\right](\eta_{p+1})\right| \quad \leq \quad {R_\eta}^{p+1} \, , \tag{2.1}$$

where

$$\begin{array}{rccc} \varphi \,:\, & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & \zeta & \mapsto & \dfrac{\overline{\zeta}}{1+|\zeta|^2} \, . \end{array}$$

Now consider for all $N \geq 1$ the Lagrange interpolation polynomial of φ:

$$L_N(\varphi)(\zeta) \quad := \quad \sum_{p=1}^{N}\left(\prod_{j=1, j\neq p}^{N}\frac{\zeta-\eta_j}{\eta_p-\eta_j}\right)\varphi\left(\eta_p\right) \, .$$

We know (see for example Lemma 4 from [12]) that $L_N(\varphi)$ can also be written as

$$\begin{aligned} L_N(\varphi)(\zeta) \quad &= \quad \sum_{p=0}^{N-1}\left(\prod_{j=1}^{p}(\zeta-\eta_j)\right)\Delta_{p,(\eta_p,\ldots,\eta_1)}[\zeta\mapsto\varphi(\zeta)]\,(\eta_{p+1}) \\ &= \quad \sum_{p=0}^{N-1}\left(\prod_{j=1}^{p}\left((\zeta-\eta_\infty)-(\eta_j-\eta_\infty)\right)\right)\Delta_{p,(\eta_p,\ldots,\eta_1)}[\zeta\mapsto\varphi(\zeta)]\,(\eta_{p+1}) \, . \end{aligned}$$

Since $\lim_{j\to\infty}\eta_j = \eta_\infty$, there is p_0 such that,

$$\forall\, j \geq p_0,\ |\eta_j - \eta_\infty| \ \leq \ \frac{1}{3R_\eta} \, .$$

Let be $V = D\left(\eta_\infty, 1/(3R_\eta)\right)$. For all $N \geq p_0$ and $\zeta \in V$, one has by (2.1)

$$\begin{aligned}
|L_N(\varphi)(\zeta)| \;\leq\; & \sum_{p=0}^{p_0-1} \prod_{j=1}^{p} \left(|\zeta| + |\eta_j|\right) \left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[\varphi]\left(\eta_{p+1}\right)\right| \\
& + \sum_{p=p_0}^{N} \left(\prod_{j=1}^{p_0} \left(|\zeta| + |\eta_j|\right) \prod_{j=p_0+1}^{p} \left(|\zeta - \eta_\infty| + |\eta_j - \eta_\infty|\right) \right) \left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[\varphi]\left(\eta_{p+1}\right)\right| \\
\leq\; & \sum_{p=0}^{p_0-1} \left(|\eta_\infty| + 1/(3R_\eta) + \max_{1\leq j\leq p_0} |\eta_j| \right)^p R_\eta^{p+1} \\
& + \sum_{p=p_0}^{N} \left(|\eta_\infty| + 1/(3R_\eta) + \max_{1\leq j\leq p_0} |\eta_j| \right)^{p_0} \left(1/(3R_\eta) + 1/(3R_\eta)\right)^{p-p_0} R_\eta^{p+1} .
\end{aligned}$$

It follows that, $\forall\, N \geq p_0$,

$$\begin{aligned}
\sup_{\zeta\in V} |L_N(\varphi)(\zeta)| \;\leq\; & \sum_{p=0}^{p_0-1} \left(|\eta_\infty| + 1/(3R_\eta) + \max_{1\leq j\leq p_0} |\eta_j| \right)^p R_\eta^{p+1} \\
& + R_\eta^{p_0+1} \left(|\eta_\infty| + 1/(3R_\eta) + \max_{1\leq j\leq p_0} |\eta_j| \right)^{p_0} \sum_{p=p_0}^{N} (2/3)^{p-p_0}
\end{aligned}$$

then

$$\begin{aligned}
\sup_{N\geq p_0} \sup_{\zeta\in V} |L_N(\varphi)(\zeta)| \;\leq\; & \sum_{p=0}^{p_0-1} \left(|\eta_\infty| + 1/(3R_\eta) + \max_{1\leq j\leq p_0} |\eta_j| \right)^p R_\eta^{p+1} \\
& + 3R_\eta^{p_0+1} \left(|\eta_\infty| + 1/(3R_\eta) + \max_{1\leq j\leq p_0} |\eta_j| \right)^{p_0} .
\end{aligned}$$

The sequence $\left(L_N(\varphi)\right)_{N\geq 1}$ of polynomials is uniformly bounded on V. Then there is a subsequence $\left(L_{N_k}(\varphi)\right)_{k\geq 1}$ that uniformly converges on any compact $K \subset V$ to a function g that is holomorphic on V.

On the other hand, one has for all $\eta_j \in V$,

$$g\left(\eta_j\right) \;=\; \lim_{k\to\infty} L_{N_k}(\varphi)\left(\eta_j\right) \;=\; \lim_{k\to\infty, N_k\geq j} L_{N_k}(\varphi)\left(\eta_j\right) \;=\; \lim_{k\to\infty, N_k\geq j} \varphi\left(\eta_j\right) \;=\; \varphi\left(\eta_j\right) ,$$

ie the (nonholomorphic) function φ coincides with g on $V \cap \{\eta_j\}_{j\geq 1}$. In particular, this yields to

$$\overline{\eta_j} \;=\; \frac{g\left(\eta_j\right)}{1 - \eta_j g(\eta_j)} . \tag{2.2}$$

On the other hand, since for all $\zeta \in \mathbb{C}$, one has

$$|\zeta\varphi(\zeta)| \;=\; \frac{|\zeta|^2}{1+|\zeta|^2} \;<\; 1 ,$$

and

$$g\left(\eta_\infty\right) \;=\; \lim_{j\to\infty} g\left(\eta_j\right) \;=\; \lim_{j\to\infty} \varphi\left(\eta_j\right) \;=\; \varphi\left(\eta_\infty\right) ,$$

it follows that $|\eta_\infty g(\eta_\infty)| = |\eta_\infty \varphi(\eta_\infty)| < 1$. Then there is an open subset $U \subset V$ such that $\eta_\infty \in U$ (then $U \in \mathcal{V}(\eta_\infty)$, ie U is a neighborhood of η_∞) and $\forall\, \zeta \in U$, $|\zeta g(\zeta)| < 1$. This allows us to define the following function by

$$\begin{array}{rcl} \widetilde{g} \,:\, U & \longrightarrow & \mathbb{C} \\ \zeta & \mapsto & \dfrac{g(\zeta)}{1-\zeta g(\zeta)}\,, \end{array}$$

that is well-defined, holomorphic and satisfies by (2.2),

$$\forall\, \eta_j \in U,\ \overline{\eta_j} \quad = \quad \widetilde{g}(\eta_j)\ ,$$

ie the set $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves.

$\surd$

Remark 2.1. Although it will not be usefull in the following, the assertion is still true if we assume that $\lim_{j\to\infty} \eta_j = \infty$. Indeed, by using the following homographic transformation (that is holomorphic from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$)

$$\begin{array}{rcl} h \,:\, \overline{\mathbb{C}} & \to & \overline{\mathbb{C}} \\ z & \mapsto & \dfrac{\overline{u}\, z+1}{z-u} \end{array}$$

(where $u \notin \overline{\{\theta_j\}_{j\geq 1}}$), and applying Lemma 12 from [12], the conclusion is the following: there is $g \in \mathcal{O}(V_0)$, where $V_0 \in \mathcal{V}(0)$ is a neighborhood of 0, such that $g(0) = 0$ and, for all η_j such that $1/\eta_j \in V_0$, one has

$$\overline{\eta_j} \quad = \quad \frac{1}{g(1/\eta_j)}\,.$$

Remark 2.2. In particular, we see from the proof that the only control of $\Delta_p\left(\overline{\zeta}/(1+|\zeta|^2)\right)$ (ie condition (1.2) with $q = 1$) is sufficient to prove the assertion. It follows that, given any convergent sequence $(\eta_j)_{j\geq 1}$ that is not real-analytically interpolated, there is a subsequence $(p_k)_{k\geq 1}$ such that, for all $k \geq 1$, one has

$$\left|\Delta_{p_k,(\eta_{p_k},\eta_{p_k-1},\ldots,\eta_{p_1})}\left[\zeta \mapsto \frac{\overline{\zeta}}{1+|\zeta|^2}\right](\eta_{p_k+1})\right| \quad \geq \quad k^{p_k}\,. \tag{2.3}$$

As a consequence, we have a constructive way to get a class of families of complex lines $\eta = \{\eta_j\}_{j\geq 1}$ whose associate interpolation formula $E_N(\cdot;\eta)$ does not converge: any convergent sequence that is not locally interpolable by real-analytic curves. The following result gives an explicit example of such families.

Corollary 2. *Let consider any sequence $(\eta_j)_{j\geq 1} \subset \mathbb{R} \cup i\mathbb{R}$, such that, $\forall\, J \geq 1$, $\exists\, j_1 \geq J$ (resp. $j_2 \geq J$), $\eta_{j_1} \in \mathbb{R} \setminus i\mathbb{R}$ (resp. $\eta_{j_2} \in i\mathbb{R} \setminus \mathbb{R}$). For example,*

$$\eta_j \quad = \quad \frac{i^j}{j}\,,\ j \geq 1\,.$$

Then the associate interpolation formula $E_N(\cdot;\eta)$ does not converge, ie there exists (at least) one function $f \in \mathcal{O}(\mathbb{C}^2)$ such that $E_N(f;\eta)$ does not converge (uniformly in any compact $K \subset \mathbb{C}^2$).

Proof. Assume that it is not the case, ie the interpolation formula $E_N(\cdot;\eta)$ is convergent. It follows by Proposition 2 that this set is locally interpolable by real-analytic curves. Since $\lim_{j\to\infty}\eta_j = 0$, there are $V \in \mathcal{V}(0)$ and $g \in \mathcal{O}(V)$ such that, for all $j \geq 1$ large enough,

$$\overline{\eta_j} \quad = \quad g(\eta_j) \ .$$

In particular, for all j even and large enough, one has

$$g(\eta_j) \ = \ \overline{\frac{(-1)^{j/2}}{j}} \ = \ \frac{(-1)^{j/2}}{j} \ = \ \eta_j \ ,$$

ie the functions g and ζ (that are both holomorphic on V) coincide on a sequence whose limit point belongs to V. It follows by the Uniqueness Principle that $g(\zeta) = \zeta$, $\forall\, \zeta \in V$.

Similarly, for all j odd and large enough, one has

$$g(\eta_j) \ = \ \overline{\frac{i\,(-1)^{(j-1)/2}}{j}} \ = \ -\frac{i\,(-1)^{(j-1)/2}}{j} \ = \ -\eta_j \ ,$$

then $g(\zeta) = -\zeta$, $\forall\, \zeta \in V$. It follows that $\zeta = -\zeta$, $\forall\, \zeta \in V$, and this is impossible.

$\surd$

2.2. **A counterexample.** In this part we deal with the following example of a sequence $(\eta_j)_{j\geq 1} \subset [0,1] + i[0,1]$, that is not locally interpolable by real-analytic curves and whose interpolation formula $E_N(\cdot;\eta)$ converges. It follows that the geometric condition (1.3) is not necessary then the assertion from Theorem 2 is not reciprocal.

The idea is the following: we define a sequence of points of the square such that the mutual distance will tend to zero but in the slowest possible way. We begin with $\eta_1 = 0$, $\eta_2 = 1$, $\eta_3 = 1+i$, $\eta_4 = i$. We find the maximal number of points in the square whose mutual distance is not smaller than 1. When it is not possible any more we add the maximal number of points whose mutual distance will be at least $1/2$, then $\eta_5 = 1/2$, $\eta_6 = i/2$, $\eta_7 = (1+i)/2$, $\eta_8 = 1 + i/2, \eta_9 = 1/2 + i$.

More generally, by induction on $r \geq 0$, we choose the maximal number of points whose mutual distance is at least $1/2^r$. This number is asymptotically 2^{2r}: this comes from the fact that the logarithm of the ε-entropy of the square (that is defined as the minimal number of balls of radius $\leq \varepsilon$ to cover it) is of order $2\ln(1/\varepsilon)$ (see for example [15]).

For all $r \geq 0$, let $\mathcal{A}_r$ be an $1/2^r$-net of the square, ie the set of the points that are at least at a distance of $1/2^r$ from each other. These points can be explicited and one has

$$\mathcal{A}_r \quad = \quad \left\{ \frac{s+it}{2^r} \,,\, 0 \leq s,t \leq 2^r \right\} \,,$$

whose cardinal is $(1+2^r)^2$. Moreover, one has the sequence of inclusions $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_r \subset \cdots$ and the sequence $\eta = (\eta_j)_{j\geq 1}$ will be defined by induction on $r \geq 0$ to get

$$\{\eta_j\}_{j\geq 1} \quad = \quad \bigcup_{r\geq 0} \mathcal{A}_r \,,$$

that will be dense in the whole square $[0,1]+i[0,1]$.

Notice that it is not important to specify in which sense we define the η_j's that belong to $\mathcal{A}_r \setminus \mathcal{A}_{r-1}$. However it is fundamental between $\mathcal{A}_{r-1}$ and $\mathcal{A}_r$ in the meaning that, if $\eta_{j_1} \in \mathcal{A}_{r-1}$ and $\eta_{j_2} \in \mathcal{A}_r \setminus \mathcal{A}_{r-1}$, then one must have $j_1 < j_2$. Equivalently, for all $r \geq 0$, the first $(2^r+1)^2$ points η_j's belong to $\mathcal{A}_r$.

Now we can give the following result as claimed in Introduction.

Proposition 3. *The sequence $(\eta_j)_{j\geq 1}$ is not locally interpolable by real-analytic curves since its topological closure has nonempty interior. Nevertheless, it makes converge its associate interpolation formula $E_N(\cdot;\eta)$, ie for all $f \in \mathcal{O}(\mathbb{C}^2)$, $E_N(f;\eta)$ converges to f uniformly on any compact $K \subset \mathbb{C}^2$.*

Before giving the proof of this result, we need to prove the following lemmas. Since we want to estimate the operator Δ_p, we begin with an auxiliary result about it.

Lemma 2. *Let $\{\theta_j\}_{j\geq 1}$ be any set of different points and h any function defined on them. Then for all $p \geq 0$, one has*

$$\Delta_{p,(\theta_p,\ldots,\theta_1)}[h](\theta_{p+1}) = \sum_{q=1}^{p+1} \frac{h(\theta_q)}{\prod_{j=1,j\neq q}^{p+1}(\theta_q-\theta_j)}.$$

Proof. We will prove this result by induction on $p \geq 0$. For $p=0$, one has

$$\Delta_{0,\emptyset}[h](\theta_1) = h(\theta_1) = \frac{h(\theta_1)}{\prod_{j\in\emptyset}(\theta_1-\theta_j)}.$$

Now assume that it is true for $p \geq 0$ and consider Δ_{p+1}. One has

$$\begin{aligned}
&\Delta_{p+1,,(\theta_{p+1},\ldots,\theta_1)}[h](\theta_{p+2}) = \\
&= \frac{\Delta_{p,(\theta_p,\ldots,\theta_1)}[h](\theta_{p+2}) - \Delta_{p,(\theta_p,\ldots,\theta_1)}[h](\theta_{p+1})}{\theta_{p+2}-\theta_{p+1}} \\
&= \sum_{q=1}^{p} \frac{h(\theta_q)}{\theta_{p+2}-\theta_{p+1}} \left(\frac{1}{(\theta_q-\theta_{p+2})\prod_{j=1,j\neq q}^{p}(\theta_q-\theta_j)} - \frac{1}{(\theta_q-\theta_{p+1})\prod_{j=1,j\neq q}^{p}(\theta_q-\theta_j)} \right) \\
&\quad + \frac{1}{\theta_{p+2}-\theta_{p+1}} \frac{h(\theta_{p+2})}{\prod_{j=1}^{p}(\theta_{p+2}-\theta_j)} + \frac{1}{\theta_{p+1}-\theta_{p+2}} \frac{h(\theta_{p+1})}{\prod_{j=1}^{p}(\theta_{p+1}-\theta_j)} \\
&= \sum_{q=1}^{p} \frac{h(\theta_q)}{(\theta_q-\theta_{p+1})(\theta_q-\theta_{p+2})\prod_{j=1,j\neq q}^{p}(\theta_q-\theta_j)} \\
&\quad + \frac{h(\theta_{p+1})}{(\theta_{p+1}-\theta_{p+2})\prod_{j=1}^{p}(\theta_{p+1}-\theta_j)} + \frac{h(\theta_{p+2})}{(\theta_{p+2}-\theta_{p+1})\prod_{j=1}^{p}(\theta_{p+2}-\theta_j)} \\
&= \sum_{q=1}^{p+2} \frac{h(\theta_q)}{\prod_{j=1,j\neq q}^{p+2}(\theta_q-\theta_j)},
\end{aligned}$$

and the induction is achieved.

$\surd$

The following lemma is an low-estimate of the products that appear on the new expression of Δ_p.

Lemma 3. *There is $p_\eta \geq 2$ such that, for all $p \geq p_\eta$ and all $q = 1, \ldots, p+1$, one has*

$$\prod_{j=1, j\neq q}^{p+1} |\eta_q - \eta_j| \quad \geq \quad \exp(-16p)\,. \tag{2.4}$$

Proof. Since $p \geq 2$, there exists $r \geq 0$ (and only one) such that

$$\left(1 + 2^{r-1}\right)^2 \; < \; p+1 \; \leq \; \left(1 + 2^r\right)^2\,. \tag{2.5}$$

In particular, the mutual distance of all the points η_j, $j = 1, \ldots, p+1$, is at least $1/2^r$ (since by construction, the first $p+1$ points η_j belong to $\mathcal{A}_r$).

Now let be $q = 1, \ldots, p+1$. There are at most 8 points η_j such that $1/2^r \leq |\eta_q - \eta_j| < 2/2^r$. Analogously, there are at most 16 points η_j such that $2/2^r \leq |\eta_q - \eta_j| < 3/2^r$. By induction on $k \geq 1$, if $\mathcal{N}(k)$ is the number of the points η_j such that $k/2^r \leq |\eta_q - \eta_j| < (k+1)/2^r$, with $k = 1, \ldots, 2^r$, then one has

$$\mathcal{N}(k) \quad \leq \quad 2 \times (2k+1) + 2 \times (2k-1) \; = \; 8k \tag{2.6}$$

(these points are in the boundary of the square of center η_q and whose lenght of the side is $2k/2^r$). It follows that

$$\begin{aligned}
\prod_{j=1, j\neq q}^{p+1} |\eta_q - \eta_j| \quad = \quad & \prod_{1/2^r \leq |\eta_j - \eta_q| < 2/2^r} |\eta_q - \eta_j| \prod_{2/2^r \leq |\eta_j - \eta_q| < 3/2^r} |\eta_q - \eta_j| \cdots \\
& \cdots \prod_{(2^r-1)/2^r \leq |\eta_j - \eta_q| < 2^r/2^r} |\eta_q - \eta_j| \prod_{2^r/2^r \leq |\eta_j - \eta_q| < (2^r+1)/2^r} |\eta_q - \eta_j| \\
& \cdots \prod_{|\eta_j - \eta_q| \geq (2^r+1)/2^r} |\eta_q - \eta_j| \\
\geq \quad & \left(\frac{1}{2^r}\right)^{\mathcal{N}(1)} \left(\frac{2}{2^r}\right)^{\mathcal{N}(2)} \cdots \left(\frac{k}{2^r}\right)^{\mathcal{N}(k)} \cdots \left(\frac{2^r}{2^r}\right)^{\mathcal{N}(r)} \times 1 \\
= \quad & \exp\left[\sum_{k=1}^{2^r} \mathcal{N}(k) \ln\left(\frac{k}{2^r}\right)\right]\,.
\end{aligned}$$

Since $k/2^r \leq 1$, $\forall\, k = 1, \ldots, 2^r$, one has by (2.6)

$$\sum_{k=1}^{2^r} \mathcal{N}(k) \ln\left(\frac{k}{2^r}\right) \quad \leq \quad \sum_{k=1}^{2^r} 8k \ln\left(\frac{k}{2^r}\right) \; = \; 2^{2r+3} \times \frac{1}{2^r} \sum_{k=1}^{2^r} \frac{k}{2^r} \ln \frac{k}{2^r}\,.$$

The last expression is the Riemann's sum of the continuous function $t \in [0,1] \mapsto t \ln t$ and whose integral is

$$\int_0^1 t \ln t dt \quad = \quad \left[\frac{t^2}{2} \ln t\right]_0^1 - \int_0^1 \frac{t}{2} dt \; = \; -\frac{1}{4}\,.$$

It follows that

$$\prod_{j=1, j\neq q}^{p+1} |\eta_q - \eta_j| \quad \geq \quad \exp\left[2^{2r+3}\left(-1/4 + \varepsilon(1/r)\right)\right]\,,$$

with $\varepsilon(1/r) \xrightarrow[1/r \to 0]{} 0$.

On the other hand, one has by (2.5)

$$p \quad \geq \quad 2^{2r-2} + 2^r \;\geq\; \frac{2^{2r}}{4}\,,$$

then

$$\prod_{j=1, j\neq q}^{p+1} |\eta_q - \eta_j| \quad \geq \quad \exp\left[8 \times 4p\left(-1/4 + \varepsilon(1/p)\right)\right] \;=\; \exp\left[-8p(1+\varepsilon(1/p))\right]$$

($\varepsilon(1/r) = \varepsilon(1/p)$ since $r \to \infty$ if and only if $p \to \infty$).

It follows that, for all $p \geq p_\eta$, one has $|\varepsilon(1/p)| \leq 1$ then for all $q = 1, \ldots, p+1$,

$$\prod_{j=1, j\neq q}^{p+1} |\eta_q - \eta_j| \quad \geq \quad \exp(-16p)\,,$$

and the proof of the lemma is achieved.

$\surd$

Then we prove the following result before giving the proof of Proposition 3.

Lemma 4. *Let h be any function defined on the set $\{\eta_j\}_{j\geq 1}$ and that is bounded:*

$$\|h\|_\infty \;:=\; \sup_{j\geq 1} |h(\eta_j)| \quad < \quad +\infty\,.$$

Then there is R_η such that, for all $p \geq 0$,

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[h]\,(\eta_{p+1})\right| \quad \leq \quad \|h\|_\infty R_\eta^p\,.$$

Proof. Let be $p \geq 0$. One has by Lemma 2

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[h]\,(\eta_{p+1})\right| \;\leq\; \sum_{q=1}^{p+1} \frac{|h(\eta_q)|}{\prod_{j=1,j\neq q}^{p+1} |\eta_q - \eta_j|} \;\leq\; \frac{(p+1)\|h\|_\infty}{\min_{1\leq q\leq p+1} \prod_{j=1,j\neq q}^{p+1} |\eta_q - \eta_j|}\,.$$

If $p \leq p_\eta - 1$, then

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[h]\,(\eta_{p+1})\right| \quad \leq \quad C_\eta \|h\|_\infty\,,$$

where

$$C_\eta \quad := \quad \frac{p_\eta}{\min_{1\leq p\leq p_\eta}\left(\min_{1\leq q\leq p+1} \prod_{j=1,j\neq q}^{p+1} |\eta_q - \eta_j|\right)}\,.$$

Otherwise $p \geq p_\eta$ (≥ 2) then one has by Lemma 3

$$\min_{1\leq q\leq p+1} \prod_{j=1,j\neq q}^{p+1} |\eta_q - \eta_j| \quad \geq \quad 1/\exp(16p)$$

thus

$$\begin{aligned}\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[h]\,(\eta_{p+1})\right| \quad &\leq \quad (p+1)\|h\|_\infty \exp(16p) \;\leq\; \exp(p)\|h\|_\infty \exp(16p)\\ &= \quad \|h\|_\infty \exp(17p)\,.\end{aligned}$$

It follows that, for all $p \geq 0$, one has

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[h]\,(\eta_{p+1})\right| \quad \leq \quad (1+C_\eta)\,\|h\|_\infty \exp(17p) \;\leq\; \|h\|_\infty R_\eta^p\,,$$

where $R_\eta := (1+C_\eta)e^{17}$ (for $p=0$, one just has $|\Delta_0(h)\,(\eta_1)| = |h(\eta_1)| \leq \|h\|_\infty$) and the proof is achieved.

$\surd$

Remark 2.3. Notice that we do not need to assume any kind of regularity for the function h else it is bounded on the set $\overline{\{\eta_j\}_{j\geq 1}}$. In particular, h can be discontinuous and have a very bad behavior on $\overline{\{\eta_j\}_{j\geq 1}} \setminus \{\eta_j\}_{j\geq 1}$.

Now we can give the proof of Proposition 3.

Proof. The set $\{\eta_j\}_{j\geq 1}$ is not locally interpolable by real-analytic curves, otherwise by Lemma 1 its topological closure would have empty interior. On the other hand, since it is bounded, in order to prove that $E_N(\cdot;\eta)$ converges, it is sufficient to show that the sequence $(\eta_j)_{j\geq 1}$ satisfies condition (1.2) from Theorem 1. For all $q \geq 0$, one has with the choice of $h(\zeta) = \left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q$,

$$\left\| \left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q \right\|_\infty \leq \left\| \frac{\overline{\zeta}}{1+|\zeta|^2} \right\|_\infty^q \leq \left\| \frac{\sqrt{1+|\zeta|^2}}{1+|\zeta|^2} \right\|_\infty^q \leq 1$$

(in particular h is bounded on $\mathbb{C}$). It follows by Lemma 4 that, for all p, $q \geq 0$, one has

$$\left| \Delta_{p,(\eta_p,\ldots,\eta_1)} \left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q \right] (\eta_{p+1}) \right| \leq \left\| \left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q \right\|_\infty \times R_\eta^p \leq R_\eta^p \leq R_\eta^{p+q},$$

ie $(\eta_j)_{j\geq 1}$ satisfies condition (1.2) from Theorem 1.

$\surd$

3. The action of the permutations of the lines (resp. subfamilies of lines): an equivalent condition

Now we deal with the problem of the action of the permutation group $\mathfrak{S}_{\mathbb{N}}$ for the convergence of the interpolation formula that is associate to the sequence $\eta = (\eta_j)_{j\geq 1}$. Although it is not specified, in the whole paper, $\mathfrak{S}_{\mathbb{N}}$ will mean the permutation group of $\mathbb{N} \setminus \{0\}$ since all the considered sequences $\eta = (\eta_j)_{j\geq 1}$ begin at $j = 1$. As said in Introduction, we will see that, for certain families, there are permutations that can change the validity of this convergence. That is why we cannot consider any more families of complex lines only as sets else also the different sequences that come from such a fixed family.

We remind that, a sequence $\eta = (\eta_j)_{j\geq 1}$ and a permutation $\sigma \in \mathfrak{S}_{\mathbb{N}}$ being given, the sequence $\sigma(\eta)$ is defined as

$$\sigma(\eta) = \left(\eta_{\sigma(1)}, \eta_{\sigma(2)}, \ldots, \eta_{\sigma(N)}, \ldots\right).$$

It is obvious that, if σ only changes a finite number of lines, it will not change the convergence of their associate formulas (since $E_N(\cdot;\eta)$ is symmetric on the N first lines η_j, $j = 1, \ldots, N$). But what will happen if it changes an infinite number of them?

3.1. **The action of the permutations can change everything.** In this part we will give the proof of Proposition 1 that we claimed in Introduction. We remind that there are two sequences of all different points such that $(\theta_j)_{j\geq 1}$ (resp. $(\kappa_j)_{j\geq 1}$) is a bounded sequence whose interpolation formula $E_N(\cdot;\theta)$ (resp. $E_N(\cdot;\kappa)$) does not converge (resp. converges) and $\operatorname{dist}\left\{\{\theta_j\}_{j\geq 1}, \{\kappa_j\}_{j\geq 1}\right\} > 0$. Finally, the sequence $(\eta_j)_{j\geq 1}$ is defined as $\eta_j = \theta_{j/2}$ (resp. $\kappa_{(j+1)/2}$) if j is even (resp. odd).

Since both sequences are bounded, so is $\eta = (\eta_j)_{j\geq 1}$, and also $\sigma(\eta) = \left(\eta_{\sigma(j)}\right)_{j\geq 1}$, for all $\sigma \in \mathfrak{S}_{\mathbb{N}}$. It follows by Theorem 1 that its associate interpolation formula will converge if and only if $\sigma(\eta)$ satisfies condition (1.2) from Theorem 1. The proof of the proposition will be a consequence of both following lemmas. The first one (resp. the second one) will give the existence of a permutation σ_1 (resp. σ_2) such that the interpolation formula $E_N(\cdot;\sigma_1(\eta))$ converges (resp. $E_N(\cdot;\sigma_2(\eta))$ does not converge).

The essential idea will be the following: the permutation σ_1 will *privilege* the sequence $(\kappa_j)_{j\geq 1}$ in the meaning that it will take a lot of them, so that $(\eta_j)_{j\geq 1}$ will have the same behavior as $(\kappa_j)_{j\geq 1}$ in terms of convergence of their associate interpolation formulas. Similarly, σ_2 will take a lot of θ_j's so that the sequence $\sigma_2(\eta)$ will make diverge its associate interpolation formula.

Lemma 5. *There is $\sigma_1 \in \mathfrak{S}_{\mathbb{N}}$ such that the sequence $\sigma_1(\eta)$ satisfies condition (1.2) from Theorem 1.*

Proof. First, one has for all $k \geq 0$ and $l \geq 1$ (see [12], Lemma 3),

$$\Delta_{k+l,(\theta_k,\ldots,\theta_1,\kappa_l,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\theta_{k+1}) \quad = $$

$$= \quad \Delta_{k,(\theta_k,\ldots,\theta_1)}\left[w \mapsto \Delta_{l,(\kappa_l,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](w)\right](\theta_{k+1})\ .$$

Since $(\kappa_j)_{j\geq 1}$ satisfies condition (1.2), there is $R_\kappa \geq 1$ such that, for all $l, q \geq 0$, one has

$$\left|\Delta_{l,(\kappa_l,\ldots,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\kappa_{l+1})\right| \quad \leq \quad R_\kappa^{l+q}\,. \tag{3.1}$$

Now we set

$$d_{\theta,\kappa} \quad := \quad \min\left\{2\,,\, \operatorname{dist}\left(\{\theta_j\}_{j\geq 1}, \{\kappa_j\}_{j\geq 1}\right)\right\}\,, \tag{3.2}$$

and we claim that, for all $q \geq 0$, $l \geq 1$ and $w \in \{\theta_j\}_{j\geq 1}$, one has

$$\left|\Delta_{l,(\kappa_l,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](w)\right| \quad \leq \quad \left(\frac{2R_\kappa}{d_{\theta.\kappa}}\right)^{l+q}\,. \tag{3.3}$$

Since for $q = 0$ it is always true with any $l \geq 1$ and $w \in \mathbb{C}$ (it will give 0), one can assume that $q \geq 1$. We prove this inequality by induction on $l \geq 1$. For $l = 1$, one has, for all $q \geq 1$ and $w \in \{\theta_j\}_{j\geq 1}$,

$$\left|\Delta_{1,\kappa_1}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](w)\right| \;\leq\; \frac{2}{|w-\kappa_1|}\sup_{\zeta\in\mathbb{C}}\left|\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right| \;\leq\; \frac{2}{d_{\theta.\kappa}} \;\leq\; \frac{2R_\kappa}{d_{\theta.\kappa}}\,,$$

since, for all $\zeta \in \mathbb{C}$, $\left|\overline{\zeta}/(1+|\zeta|^2)\right| \leq 1$.

Now if we assume that it is true for $l \geq 1$, for all $q \geq 1$ and all $w \in \{\theta_j\}_{j\geq 1}$, one has induction hypothesis, (3.1) and (3.2)

$$\begin{aligned}
&\left|\Delta_{l+1,(\kappa_{l+1},\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](w)\right| \quad \leq \\
&\qquad \leq \quad \frac{\left|\Delta_{l,(\kappa_l,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](w)\right| + \left|\Delta_{l,(\kappa_l,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\kappa_{l+1})\right|}{|w-\kappa_{l+1}|} \\
&\qquad \leq \quad \frac{(2R_\kappa/d_{\theta.\kappa})^{l+q} + R_\kappa^{l+q}}{d_{\theta.\kappa}} \;\leq\; \frac{(2R_\kappa/d_{\theta.\kappa})^{l+q} + (2R_\kappa/d_{\theta.\kappa})^{l+q}}{d_{\theta.\kappa}} \\
&\qquad = \quad (2R_\kappa/d_{\theta.\kappa})^{l+q+1}\ ,
\end{aligned}$$

and this proves (3.3) for $l+1$, all $q \geq 1$ and all $w \in \{\theta_j\}_{j\geq 1}$.

Now we define, for all $k \geq 1$ and all $w \in \{\theta_j\}_{j\geq k+1}$,

$$\phi_k(w) \quad := \quad \min\left\{1\,,\,\left[\min_{1\leq j\leq k}|w-\theta_j|\right]\,,\,\left[\min_{1\leq i<j\leq k}|\theta_j-\theta_i|\right]\right\} \;>\; 0 \tag{3.4}$$

(for $k=0$, we agree that $\phi_0(w) := 1$, $\forall\, w \in \{\theta_j\}_{j\geq 1}$). In particular, one has for all $k \geq 1$ and $w \in \{\theta_j\}_{j\geq k+1}$,

$$\phi_k(w) \quad \leq \quad \min\left\{|w-\theta_k|\ ,\ \phi_{k-1}(w)\,,\ \phi_{k-1}(\theta_k)\right\}\ . \tag{3.5}$$

Then we claim that, for all $k,l,q \geq 0$ and all $w \in \{\theta_j\}_{j\geq k+1}$, one has

$$\left|\Delta_{k+l,(\theta_k,\ldots,\theta_1,\kappa_l,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](w)\right| \;\leq\; \left(\frac{2}{\phi_k(w)}\right)^k\left(\frac{2R_\kappa}{d_{\theta.\kappa}}\right)^{l+q}\ . \tag{3.6}$$

We prove this inequality by induction on $k \geq 0$. For $k=0$, it is a consequence of (3.3). Now if $k \geq 1$, one has by induction and (3.5)

$$\begin{aligned}
&\left|\Delta_{k+l,(\theta_k,\ldots,\theta_1,\kappa_l,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](w)\right| \quad \leq \\
&\quad \leq \quad \frac{\left|\Delta_{k+l-1,(\theta_{k-1},\ldots,\theta_1,\kappa_l,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](w)\right| + \left|\Delta_{k+l-1,(\theta_{k-1},\ldots,\theta_1,\kappa_l,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\theta_k)\right|}{|w-\theta_k|} \\
&\quad \leq \quad 2^{k-1}\left(\frac{2R_\kappa}{d_{\theta.\kappa}}\right)^{l+q}\frac{1}{|w-\theta_k|}\left(\frac{1}{\phi_{k-1}(w)^{k-1}} + \frac{1}{\phi_{k-1}(\theta_k)^{k-1}}\right) \\
&\quad \leq \quad 2^{k-1}\left(\frac{2R_\kappa}{d_{\theta.\kappa}}\right)^{l+q}\frac{1}{\phi_k(w)}\frac{2}{\phi_k(w)^{k-1}} \\
&\quad = \quad \left(\frac{2}{\phi_k(w)}\right)^k\left(\frac{2R_\kappa}{d_{\theta.\kappa}}\right)^{l+q}\ ,
\end{aligned}$$

and this proves (3.6) for k, for all $l,q \geq 0$ and all $w \in \{\theta_j\}_{j\geq k+1}$. In particular, it follows that, for all $k,l,q \geq 0$,

$$\left|\Delta_{k+l,(\theta_k,\ldots,\theta_1,\kappa_l,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\theta_{k+1})\right| \;\leq\; \left(\frac{2}{\phi_k(\theta_{k+1})}\right)^k\left(\frac{2R_\kappa}{d_{\theta.\kappa}}\right)^{l+q}\ . \tag{3.7}$$

Now for all $k \geq 0$, we define l_k the smallest integer $l \geq 1$ such that

$$l \quad \geq \quad k \max\left\{1\,,\; \frac{\ln\left(\dfrac{d_{\theta,\kappa}}{2\phi_k(\theta_{k+1})R_\kappa}\right)}{\ln 2}\right\} \tag{3.8}$$

and such that the sequence $(l_k)_{k\geq 0}$ is increasing. It follows that, for all $k, q \geq 0$ and $l \geq l_k$, one has

$$\left(\frac{1}{\phi_k(\theta_{k+1})}\right)^k \quad \leq \quad 2^l \left(\frac{2R_\kappa}{d_{\theta,\kappa}}\right)^k\,,$$

then

$$\begin{aligned}\left(\frac{4R_\kappa}{d_{\theta,\kappa}}\right)^{k+l+q} \quad &= \quad 2^{k+l+q}\left(\frac{2R_\kappa}{d_{\theta,\kappa}}\right)^k \left(\frac{2R_\kappa}{d_{\theta,\kappa}}\right)^{l+q} \;\geq\; 2^k 2^l \left(\frac{2R_\kappa}{d_{\theta,\kappa}}\right)^k \left(\frac{2R_\kappa}{d_{\theta,\kappa}}\right)^{l+q}\\ &\geq \quad 2^k\left(\frac{1}{\phi_k(\theta_{k+1})}\right)^k \left(\frac{2R_\kappa}{d_{\theta,\kappa}}\right)^{l+q}\,,\end{aligned}$$

thus with (3.7) this inequality yields to

$$\left|\Delta_{k+l,(\theta_k,\ldots,\theta_1,\kappa_l,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\theta_{k+1})\right| \quad \leq \quad R_{\theta,\kappa}^{k+l+q}\,, \tag{3.9}$$

where

$$R_{\theta,\kappa} \quad := \quad \frac{4R_\kappa}{d_{\theta,\kappa}}\,. \tag{3.10}$$

Now we will construct the permutation σ_1 as follows: for all $j = 1, \ldots, l_1$, we set $\eta_{\sigma_1(j)} := \kappa_j$ and for all $k \geq 1$, we set

$$\eta_{\sigma_1(j)} \quad := \quad \begin{cases}\theta_k \text{ if } j = l_k + k\,,\\ \kappa_{j-k} \text{ if } j = l_k + k + 1, \ldots, l_{k+1} + k\,.\end{cases} \tag{3.11}$$

Notice that all the θ_j's are reached exactly once by the sequence $\left(\eta_{\sigma_1(j)}\right)_{j\geq 1}$. On the other hand, for all $k \geq 1$, one has $l_k + k + 1 \leq l_{k+1} + k$ since $(l_k)_{k\geq 1}$ is increasing (then there is at least one κ_j between l_k+k+1 and $l_{k+1}+k$) and by construction, all the κ_j's are reached for $j = l_k+k+1, \ldots, l_{k+1}+k$. Finally, since $\eta_{\sigma_1(l_{k+1}+k)} = \kappa_{l_{k+1}}$ and $\eta_{\sigma_1(l_{k+1}+(k+1)+1)} = \kappa_{l_{k+1}+1}$, one can deduce by induction on $k \geq 1$ that all the κ_j's are reached (and exactly once) by the sequence $\left(\eta_{\sigma_1(j)}\right)_{j\geq 1}$. This proves that σ_1 is a permutation.

To finish the proof, we just need to prove that the sequence $\sigma_1(\eta)$ satisfies the condition (1.2) from Theorem 1 with $R_{\theta,\kappa}$. First, for all $p = 1, \ldots, l_1 - 1$ and all $q \geq 0$, one has by (3.1)

$$\begin{aligned}\left|\Delta_{p,\left(\eta_{\sigma_1(p)},\ldots,\eta_{\sigma_1(1)}\right)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right]\left(\eta_{\sigma_1(p+1)}\right)\right| \quad &= \quad \left|\Delta_{p,(\kappa_p,\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\kappa_{p+1})\right|\\ &\leq \quad R_\kappa^{p+q} \;\leq\; R_{\theta,\kappa}^{p+q}\,,\end{aligned}$$

since by (3.2) and (3.10) one has $R_{\theta,\kappa} \geq R_\kappa$ (and the inequality is obvious for $p=0$ and $q \geq 0$). Otherwise, one has $p+1 \geq l_1+1$ then there is $k \geq 1$ such that $l_k + k \leq p+1 \leq l_{k+1} - k$. By (3.9) and Lemma 7 from [12], it follows that one has

$$\left|\Delta_{p,\left(\eta_{\sigma_1(p)},\ldots,\eta_{\sigma_1(1)}\right)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right]\left(\eta_{\sigma_1(p+1)}\right)\right| =$$

$$= \left|\Delta_{p,(\theta_{k-1},\ldots,\theta_1,\kappa_{p-k+1},\ldots,\kappa_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\theta_k)\right| \leq R_{\theta,\kappa}^{(k-1)+(p-k+1)+q} = R_{\theta,\kappa}^{p+q}$$

(since $p-k+1 \geq l_k \geq l_{k-1}$).

$\surd$

Now we will prove the second part of Proposition 1: there is $\sigma_2 \in \mathfrak{S}_{\mathbb{N}}$ such that the interpolation formula $E_N(\cdot;\sigma_2(\eta))$ does not converge.

Lemma 6. *There is $\sigma_2 \in \mathfrak{S}_{\mathbb{N}}$ such that the sequence $\sigma_2(\eta)$ does not satisfies condition (1.2) from Theorem 1.*

Proof. We begin with considering the bounded sequence θ. Since its associate interpolation formula $E_N(\cdot;\theta)$ does not converge, it follows by Theorem 1 that θ does not satisfies the criterion (1.2), ie for all $r \geq 1$ there are $p_r, q_r \geq 1$ (that one can choose such that the sequence $(p_r+q_r)_{r\geq 1}$ is increasing) that satisfy:

$$\left|\Delta_{p_r,(\theta_{p_r},\theta_{p_r-1},\ldots,\theta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q_r}\right](\theta_{p_r+1})\right| \geq r^{p_r+q_r}\,. \tag{3.12}$$

We fix p_1 and we define the new sequence

$$\theta^{(1)} := (\theta_1,\ldots,\theta_{p_1},\theta_{p_1+1},\kappa_1,\theta_{p_1+2},\ldots)\,.$$

Then one still has:

$$\left|\Delta_{p_1,\left(\theta^{(1)}_{p_1},\ldots,\theta^{(1)}_1\right)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q_1}\right]\left(\theta^{(1)}_{p_1+1}\right)\right| = \left|\Delta_{p_1,(\theta_{p_1},\ldots,\theta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q_1}\right](\theta_{p_1+1})\right| \geq 1^{p_1+q_1}\,.$$

Now assume having constructed for all $n \geq 1$ the sequence $\theta^{(n)}$ such that:

- $\left\{\theta^{(n)}_j\right\}_{j\geq 1} = \{\theta_j\}_{j\geq 1}\bigcup\{\kappa_1,\ldots,\kappa_n\}$;
- there are $p_1,p_2,\ldots,p_n \geq 1$ and $q_1,q_2,\ldots,q_n \geq 1$ such that, for all $r = 1,\ldots,n$, one has $p_r \geq p_{r-1}+2$ and
$$\left|\Delta_{p_r,\left(\theta^{(n)}_{p_r},\theta^{(n)}_{p_r-1},\ldots,\theta^{(n)}_1\right)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q}\right]\left(\theta^{(n)}_{p_r+1}\right)\right| \geq r^{p_r+q_r}\,.$$
- For all $j = 1,\ldots,p_n+1$, $\theta^{(n)}_j = \theta^{(n-1)}_j$.

This assertion will be proved by induction on $n \geq 1$ (the case $n=1$ has already been proved above). The sequence $\theta^{(n)}$ is such that its associate interpolation formula $E_N\left(\cdot;\theta^{(n)}\right)$ does not converge. Indeed, assume that it is not the case. Then by Corollary 6 (whose proof will be given in the next Section), the new

sequence $\widetilde{\theta}$ that is $\theta^{(n)}$ minus the points $\kappa_1, \ldots, \kappa_n$, would still make converge its associate interpolation formula. But it is impossible since this new sequence $\widetilde{\theta}$ is exactly θ whose associate interpolation formula does not converge.

It follows that there are an infinite number of $r \geq n+1$ for which there are p'_r, q'_r that satisfy (3.12) with the sequence $\theta^{(n)}$. Then exactly one of two things can happen.

(1) The set $\{p'_r\}_{r\geq n+1}$ is finite then let be

$$p_{max} \quad := \quad \max\left\{p_1, \ldots, p_n, \max_{r\geq n+1} p'_r\right\} .$$

And for all $r \geq n+1$, we set q_r to be this q'_r. Then we define σ_2 as follows:

$$\eta_{\sigma_2(j)} \ := \ \begin{cases} \theta_j^{(n)} \text{ if } 1 \leq j \leq p_{max}+1\,, \\ \theta_{(j+p_{max})/2-n+1} \text{ if } j \geq p_{max}+2 \text{ and } j-p_{max} \text{ is even,} \\ \kappa_{(j-p_{max}-1)/2+n} \text{ if } j \geq p_{max}+2 \text{ and } j-p_{max} \text{ is odd.} \end{cases} \tag{3.13}$$

Then σ_2 is a permutation since all the θ_j's (resp. κ_j's) are reached exactly once: all the $p_{max}-n+1$ first θ_j's (resp. n first κ_j's) are already reached by the $p_{max}+1$ first $\eta_{\sigma_2(j)}$; and the remaining θ_j's (resp. κ_j's) from $p_{max}-n+2$ (resp. $n+1$) are reached by the $\eta_{\sigma_2(j)}$ from $p_{max}+2$ (resp. $p_{max}+3$).

On the other hand, one has by construction $\eta_{\sigma_2(j)} = \theta_j^{(n)}$, for all $j \leq p_{max}+1$. By induction hypothesis and by (3.12), it follows that, for all $r \geq 1$, there are p_r, q_r (with $p_r \leq p_{max}$ and q_r being the associate q'_r)) such that

$$\left|\Delta_{p_r,\left(\eta_{\sigma_2(p_r)},\ldots,\eta_{\sigma_2(1)}\right)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q_r}\right]\left(\eta_{\sigma_2(p_r+1)}\right)\right| \ =$$

$$= \ \left|\Delta_{p_r,\left(\theta_{p_r}^{(n)},\ldots,\theta_1^{(n)}\right)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q_r}\right]\left(\theta_{p_r+1}^{(n)}\right)\right| \ \geq \ r^{p_r+q_r}\,,$$

ie the (bounded) sequence $\sigma_2(\eta)$ does not satisfy condition (1.2) from Theorem 1. It follows that the associate interpolation formula $E_N(\cdot;\sigma_2(\eta))$ does not converge and the lemma is already proved.

(2) Otherwise, the set $\{p'_r\}_{r\geq n+1}$ is infinite. Then one can choose p'_r large enough so that $p'_r \geq p_n+2$ and $\{\kappa_1, \ldots, \kappa_n\} \subset \left\{\theta_j^{(n)}\right\}_{1\leq j\leq p'_r-1}$. We set p_{n+1} (resp. q_{n+1}) to be this p'_r (resp. q'_r) and one still has (since $r \geq n+1$)

$$\left|\Delta_{p_{n+1},\left(\theta_{p_{n+1}}^{(n)},\ldots,\theta_1^{(n)}\right)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q_{n+1}}\right]\left(\theta_{p_{n+1}+1}^{(n)}\right)\right| \ \geq \ (n+1)^{p_{n+1}+q_{n+1}}\,.$$

Then we can define the sequence $\left(\theta_j^{(n+1)}\right)_{j\geq 1}$ as follows:

$$\theta_j^{(n+1)} \ = \ \begin{cases} \theta_j^{(n)} \text{ if } j \leq p_{n+1}+1\,, \\ \kappa_{n+1} \text{ if } j = p_{n+1}+2\,, \\ \theta_{j-1}^{(n)} \text{ if } j \geq p_{n+1}+3\,. \end{cases}$$

Since $\theta_j^{(n+1)} = \theta_j^{(n)}$, $\forall\, j = 1, \ldots, p_{n+1}+1$, one still has, for all $r = 1, \ldots, n+1$,

$$\left| \Delta_{p_r, \left(\theta_{p_r}^{(n+1)}, \ldots, \theta_1^{(n+1)}\right)} \left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2} \right)^{q_r} \right] \left(\theta_{p_r+1}^{(n+1)} \right) \right| \quad \geq \quad r^{p_r+q_r}\,.$$

Next, one has by construction

$$\left\{ \theta_j^{(m+1)} \right\}_{j\geq 1} \;=\; \left\{ \theta_j^{(m)} \right\}_{j\geq 1} \bigcup \{\kappa_{m+1}\} \;=\; \{\theta_j\}_{j\geq 1} \bigcup \{\kappa_1, \ldots, \kappa_m, \kappa_{m+1}\}\,.$$

This finally proves the induction for $n+1$.

The conclusion is following: either there exists $n \geq 1$ such that the first case happens and the proof is achieved; either one can construct by induction on $n \geq 1$ a sequence of sequences

$$\left(\left(\theta_j^{(n)} \right)_{j\geq 1} \right)_{n\geq 1}$$

that satisfy the above conditions. This allows us to define σ_2 by the diagonal sequence of the $\theta^{(n)}$'s: for all $j \geq 1$, let n_j be the smallest $n \geq 1$ such that $j \leq p_n + 1$ (such an n exists since $p_n \to \infty$ as $n \to \infty$). We set

$$\eta_{\sigma_2(j)} \quad := \quad \theta_j^{(n_j)}$$

(notice that one still has $\eta_{\sigma_2(j)} = \theta_j^{(n)}$, for all $n \geq n_j$). Then σ_2 is a permutation: indeed, all the κ_j's are reached by the $\eta_{\sigma_2(p_j+2)}$; on the other hand, all the θ_j's are reached by the $\eta_{\sigma_2(s)}$, $p_n + 3 \leq s \leq p_{n+1} + 3$, $n \geq 1$ (by construction, $\theta_{p_n+1}^{(n)} = \theta_{p_n+1-n+1} = \theta_{p_n-n+2}$ and $\theta_{p_n+3}^{(n+1)} = \theta_{p_n+3-1}^{(n)} = \theta_{p_n+2}^{(n)} = \theta_{p_n-n+3}$); finally, the condition $p_{n+1} \geq p_n + 2$ makes sure that at least one of following θ_j's is reached between $p_n + 3$ and $p_{n+1} + 1$.

Finally, for all $r \geq 1$, there are p_r and q_r such that

$$\left| \Delta_{p_r, \left(\eta_{\sigma_2(p_r)}, \ldots, \eta_{\sigma_2(1)}\right)} \left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2} \right)^{q_r} \right] \left(\eta_{\sigma_2(p_r+1)} \right) \right| \quad = $$

$$= \quad \left| \Delta_{p_r, \left(\theta_{p_r}^{(r)}, \ldots, \theta_1^{(r)}\right)} \left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2} \right)^{q_r} \right] \left(\theta_{p_r+1}^{(r)} \right) \right| \;\geq\; r^{p_r+q_r}\,,$$

ie the (bounded) sequence $\sigma_2(\eta)$ does not satisfy condition (1.2) from Theorem 1. It follows that its associate interpolation formula does not converge and the lemma is proved.

$\surd$

As a consequence, we get the following example that we mentioned in Introduction.

Corollary 3. *The choice of* $(\theta_j)_{j\geq 1}$ *and* $(\kappa_j)_{j\geq 1}$*, where*

$$\begin{cases} \theta_j \;=\; i^j/j\,, \\ \kappa_j \;=\; 3 + \sin(j)\,, \end{cases} \qquad \forall\, j \geq 1\,,$$

satisfies the conditions of Proposition 1.

It follows that, if $\eta = (\eta_j)_{j\geq 1}$ is the associate defined sequence, then there is σ_1 (resp. σ_2) such that the formula $E_N(\cdot;\sigma_1(\eta))$ converges (resp. $E_N(\cdot;\sigma_2(\eta))$ does not).

Proof. First, we know from Corollary 2 that $(\theta_j)_{j\geq 1}$ is a convergent sequence that is not locally interpolable by real-analytic curves. It follows that its associate interpolation formula does not converge.

Second, the set $\{\kappa_j\}_{j\geq 1}$ is bounded and locally interpolable by real-analytic curves since it is a bounded subset of $\mathbb{R}$. It follows by Theorem 2 that the sequence $(\kappa_j)_{j\geq 1}$ makes converge its associate interpolation formula.

Next, all the κ_j's are different since π is not rational (and so are the θ_j's).

Finally, for all $j, k \geq 1$, one has

$$|\theta_j - \kappa_k| \;\geq\; |\kappa_k| - |\theta_j| \;\geq\; (3-1) - 1 \;=\; 1 \;>\; 0\,.$$

$\surd$

We also deduce as another application a class of counterexamples for the reciprocal sense of Theorem 2.

Corollary 4. *All these examples of $\sigma_1(\eta)$ are sets that are not locally interpolable by real-analytical curves since $\{\theta_j\}_{j\geq 1}$ is even not. Nevertheless, all of their associate interpolation formula $E_N(\cdot;\sigma_1(\eta))$ converge.*

3.2. On the problem of the subsequences of $(\eta_j)_{j\geq 1}$: a first result.

In this part we deal with the problem of the stability by subsequences for the convergence of $E_N(\cdot;\eta)$. As said in Introduction, the action of taking subsequences can change everything. As an application of Corollary 3, the sequence θ can be taken back from the sequence $\sigma_1(\eta)$ whose interpolation formula converges. Similarly, the sequence κ can be taken back from the sequence $\sigma_2(\eta)$ whose interpolation formula does not converge. This yields to the following result.

Corollary 5. *There are sequences $\eta_c = \left(\eta_j^c\right)_{j\geq 1}$ such that:*

- *the associate interpolation formula $E_N(\cdot;\eta_c)$ converges;*
- *there is a subsequence $\eta_c' = \left(\eta_{j_k}^c\right)_{k\geq 1}$ whose interpolation formula $E_N(\cdot;\eta_c')$ does not converge.*

Similarly, there are sequences $\eta_d = \left(\eta_j^d\right)_{j\geq 1}$ such that:

- *the associate interpolation formula $E_N(\cdot;\eta_d)$ does not converge;*
- *there is a subsequence $\eta_d' = \left(\eta_{j_k}^d\right)_{k\geq 1}$ whose interpolation formula $E_N(\cdot;\eta_d')$ converges.*

Now that we know that generally there is no link between the validity of the convergence of $E_N(\cdot;\eta)$ and the one of any given subsequence, the following problem is to find conditions for the stability of the convergence of $E_N(\cdot;\eta)$ by subsequences. This yields to the principal result of the next part.

3.3. **A sufficient criterion for the stability of the convergence by subsequences.** As claimed in Introduction, we give here the proof of the following result that will be usefull in order to prove Theorems 3 and 4.

Proposition 4. *Let $\eta = (\eta_j)_{j\geq 1}$ be a sequence such that its associate interpolation formula converges, also uniformly on any compact subset of holomorphic functions.*

On the other hand, let $\eta' := (\eta_{j_s})_{s\geq 1}$ be any subsequence that satisfies the following conditions:

- $\|\eta'\|_\infty := \sup_{s\geq 1} |\eta_{j_s}| < +\infty$.
- *There are $C, D > 0$ such that, for all $N \geq 1$, one has*
$$\mathit{card}\{\eta(N) \setminus \eta'\} \;=\; \left\{\eta(N) \setminus \{\eta_{j_s}\}_{s\geq 1}\right\} \;\geq\; \frac{N-D}{C},$$
where $\eta(N) := \{\eta_1, \eta_2, \ldots, \eta_{N-1}, \eta_N\}$.

Then the new sequence $\eta \setminus \eta' = (\eta_j)_{j\geq 1} \setminus (\eta_{j_s})_{s\geq 1}$ (whose indexing is the natural one) still makes converge its associate interpolation formula.

Remark 3.1. In the statement we assume a little bit more than the simple convergence of $E_N(f;\cdot)$, f by f, else a uniform convergence on any compact subset of holomorphic functions (ie on closed subsets $\mathcal{K} \subset \mathcal{O}\left(\mathbb{C}^2\right)$ such that, for all compact subset $K \subset \mathbb{C}^2$, one has
$$\sup_{f\in\mathcal{K}} \sup_{z\in K} |f(z)| \quad < \quad +\infty \ ).$$
Nevertheless, it is still a legitimate condition since, in case of convergence of $E_N(f,\eta)$ to f, we get in addition a uniform convergence on compact subsets of entire functions: for all compact subsets $\mathcal{K} \subset \mathcal{O}\left(\mathbb{C}^2\right)$ and $K \subset \mathbb{C}^2$, one has
$$\sup_{f\in\mathcal{K}} \sup_{z\in K} |E_N(f;\eta)(z) - f(z)| \xrightarrow[N\to\infty]{} 0 \tag{3.14}$$
(see for example Corollary 9 from Section 4 below and also in [12] the proofs of the above Theorems 1 and 2 given in Introduction).

Before giving the proof of this proposition, we need both preliminar results. First, we remind the following one, whose proof is given in [12], and that gives an equivalence between the convergence of $E_N(\cdot;\eta)$ and the one of another analogous formula $R_N(\cdot;\eta)$.

Lemma 7. *The sequence $\eta = (\eta_j)_{j\geq 1}$ being given, one has, for any $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, for all $N \geq 1$ and $z \in \mathbb{C}^2$,*
$$f(z) \quad = \quad E_N(f;\eta)(z) - R_N(f;\eta)(z) + \sum_{k+l\geq N} a_{k,l} z_1^k z_2^l\,,$$
where $f(z) = \sum_{k+l\geq 0} a_{k,l} z_1^k z_2^l$ is its Taylor expansion and
$$R_N(f;\eta)(z) \quad = \quad \sum_{p=1}^{N} \left(\prod_{j=1, j\neq p}^{N} \frac{z_1 - \eta_j z_2}{\eta_p - \eta_j} \right) \sum_{k+l\geq N} a_{k,l} \eta_p^k \left(\frac{z_2 + \overline{\eta_p}\, z_1}{1 + |\eta_p|^2} \right)^{k+l-N+1} .$$

Since the remaining sum $\sum_{k+l\geq N} a_{k,l} z_1^k z_2^l$ always converges to 0 uniformly on any compact (also uniformly on any compact of entire functions), it follows that $E_N(f;\eta)$ converges to f uniformly on any compact subset, if and only if so does $R_N(f;\eta)$ to 0.

Now we can give the proof of the second preliminar result that will be usefull in order to prove Proposition 4.

Lemma 8. *Let be $\eta = (\eta_j)_{j\geq 1}$ and $f \in \mathcal{O}\left(\mathbb{C}^2\right)$. Then for all $N \geq 1$ and $z \in \mathbb{C}^2$, one has*

$$\begin{aligned} R_N(f;\eta)(z) \quad = \quad & \sum_{k+l\geq N} a_{k,l} z_1^k z_2^l \; - \; \prod_{j=1}^{N} (z_1 - \eta_j z_2) \times \\ & \times \frac{1}{2i\pi} \int_{\zeta\in\mathbb{C}} \frac{\sum_{k+l\geq N}(k+l-N+1)a_{k,l}\zeta^k \left(\dfrac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{k+l-N+1} d\overline{\zeta}\wedge d\zeta}{\prod_{j=1}^N (\zeta-\eta_j)\ (1+|\zeta|^2)^2}, \end{aligned}$$

where $f(z) = \sum_{k,j\geq 0} a_{k,l} z_1^k z_2^l$ is the Taylor expansion of f.

Proof. Let be $k+l \geq N$ and $z \in \mathbb{C}^2$ such that $z_2 \neq 0$ and $z_1/z_2 \neq \eta_j$, $\forall\, j = 1,\ldots,N$. For all $R > |z_1/z_2|, \max_{1\leq j\leq N} |\eta_j|$, one has by the Cauchy-Green-Pompeiu formula

$$\begin{aligned} \prod_{j=1}^{N}(z_1/z_2-\eta_j)\frac{1}{2i\pi}\int_{|\zeta|=R} \frac{\zeta^k}{\prod_{j=1}^N(\zeta-\eta_j)} \frac{1}{\zeta - z_1/z_2} \left(\frac{1+\overline{\zeta}z_1/z_2}{1+|\zeta|^2}\right)^{k+l-N+1} d\zeta \quad = & \\ = \quad \left(\frac{z_1}{z_2}\right)^k - \sum_{p=1}^{N} \prod_{j=1,j\neq p}^{N} \left(\frac{z_1/z_2-\eta_j}{\eta_p-\eta_j}\right)\eta_p^k \left(\frac{1+\overline{\eta_p}\,z_1/z_2}{1+|\eta_p|^2}\right)^{k+l-N+1} & \\ + \prod_{j=1}^{N}(z_1/z_2-\eta_j)\frac{1}{2i\pi}\int_{|\zeta|<R} \frac{\partial}{\partial\overline{\zeta}}\left[\frac{\zeta^k}{(\zeta-z_1/z_2)\prod_{j=1}^N(\zeta-\eta_j)}\left(\frac{1+\overline{\zeta}z_1/z_2}{1+|\zeta|^2}\right)^{k+l-N+1}\right] d\overline{\zeta}\wedge d\zeta\,. & \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{p=1}^{N} \prod_{j=1,j\neq p}^{N} \left(\frac{z_1-\eta_j z_2}{\eta_p-\eta_j}\right)\eta_p^k \left(\frac{z_2+\overline{\eta_p}\,z_1}{1+|\eta_p|^2}\right)^{k+l-N+1} \quad = & \\ = \quad z_2^{k+l}\sum_{p=1}^{N} \prod_{j=1,j\neq p}^{N} \left(\frac{z_1/z_2-\eta_j}{\eta_p-\eta_j}\right)\eta_p^k \left(\frac{1+\overline{\eta_p}\,z_1/z_2}{1+|\eta_p|^2}\right)^{k+l-N+1} & \\ = \quad z_1^k z_2^l - z_2^{k+l}\prod_{j=1}^{N}(z_1/z_2-\eta_j)\frac{1}{2i\pi}\int_{|\zeta|=R} \frac{\zeta^k}{\prod_{j=1}^N(\zeta-\eta_j)} \frac{1}{\zeta - z_1/z_2} \left(\frac{1+\overline{\zeta}z_1/z_2}{1+|\zeta|^2}\right)^{k+l-N+1} d\zeta & \\ + z_2^{k+l}\prod_{j=1}^{N}(z_1/z_2-\eta_j)\frac{1}{2i\pi}\int_{|\zeta|<R} \frac{\zeta^k}{(\zeta-z_1/z_2)\prod_{j=1}^N(\zeta-\eta_j)} \times & \\ \times (k+l-N+1)\left(\frac{1+\overline{\zeta}z_1/z_2}{1+|\zeta|^2}\right)^{k+l-N} \frac{\partial}{\partial\overline{\zeta}}\left[\frac{1+\overline{\zeta}z_1/z_2}{1+\zeta\overline{\zeta}}\right] d\overline{\zeta}\wedge d\zeta & \\ = \quad z_1^k z_2^l - z_2^{k+l}\prod_{j=1}^{N}(z_1/z_2-\eta_j)\frac{1}{2i\pi}\int_{|\zeta|=R} \frac{\zeta^k}{\prod_{j=1}^N(\zeta-\eta_j)} \frac{1}{\zeta - z_1/z_2} \left(\frac{1+\overline{\zeta}z_1/z_2}{1+|\zeta|^2}\right)^{k+l-N+1} d\zeta & \\ - \prod_{j=1}^{N}(z_1-\eta_j z_2)\frac{1}{2i\pi}\int_{|\zeta|<R} \frac{(k+l-N+1)\zeta^k}{\prod_{j=1}^N(\zeta-\eta_j)} \left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{k+l-N} \frac{d\overline{\zeta}\wedge d\zeta}{(1+|\zeta|^2)^2}\,. & \end{aligned}$$

Since

$$\deg_X \left(\frac{X^k}{\prod_{j=1}^N (X-\eta_j)} \left(\frac{1+Xz_1/z_2}{1+X^2} \right)^{k+l-N+1} \right) \ \leq \ k-N-(k+l-N+1) \ \leq \ -1\,,$$

one can consider the limit as $R \to +\infty$, then

$$\sum_{p=1}^N \prod_{j=1, j\neq p}^N \left(\frac{z_1-\eta_j z_2}{\eta_p-\eta_j} \right) \eta_p^k \left(\frac{z_2+\overline{\eta_p}\, z_1}{1+|\eta_p|^2} \right)^{k+l-N+1} \quad =$$

$$= \quad z_1^k z_2^l - \prod_{j=1}^N (z_1-\eta_j z_2) \frac{1}{2i\pi} \int_{\mathbb{C}} \frac{(k+l-N+1)\zeta^k}{\prod_{j=1}^N(\zeta-\eta_j)} \left(\frac{z_2+\overline{\zeta} z_1}{1+|\zeta|^2} \right)^{k+l-N} \frac{d\overline{\zeta}\wedge d\zeta}{(1+|\zeta|^2)^2}\,.$$

On the other hand, this equality is true for all fixed $z \in \mathbb{C}^2$ such that $z_2 \neq 0$ and $z_1/z_2 \neq \eta_j$, $\forall\, j=1,\dots,N$. By continuity, it can be extended for all $z \in \mathbb{C}^2$.

In order to prove the lemma, we want to justify the switch between series and integral. Since there is no difficulty to prove it for the part of the integral with bounded ζ, one can assume that $|\zeta| \geq R$, where R will be large enough. Let $K \subset \mathbb{C}$ be a compact subset. By the Cauchy-Schwarz inequality, one has for all $z \in K$ and $\zeta \in \mathbb{C}$,

$$\left| \frac{z_2+\overline{\zeta} z_1}{\sqrt{1+|\zeta|^2}} \right| \ \leq \ \frac{\|z\|\sqrt{1+|\zeta|^2}}{\sqrt{1+|\zeta|^2}} \ \leq \ \|z\|_K\,,$$

where

$$\|z\| \quad := \quad \sqrt{|z_1|^2+|z_1|^2} \tag{3.15}$$

and

$$\|z\|_K \quad := \quad \sup_{z\in K} \|z\|\,. \tag{3.16}$$

On the other hand, if we analogously define

$$\|f\|_R \quad := \quad \sup_{|z_1|,|z_2|\leq R} |f(z_1,z_2)|\,, \tag{3.17}$$

then one also has by the Cauchy inequalities, for all $k,l \geq 0$ and $R>0$,

$$|a_{k,l}| \quad \leq \quad \frac{\|f\|_R}{R^{k+l}}\,. \tag{3.18}$$

It follows that, for all $|\zeta| \geq 1$,

$$\sum_{k+l\geq N} (k+l-N+1) \left| a_{k,l}\zeta^k \right| \left| \frac{z_2+\overline{\zeta} z_1}{1+|\zeta|^2} \right|^{k+l-N} \quad \leq$$

$$
\begin{aligned}
&\leq \sum_{k+l\geq N} (k+l-N+1)\left|a_{k,l}\zeta^k\right| \left(\frac{\|z\|_K}{\sqrt{1+|\zeta|^2}}\right)^{k+l-N} \\
&= \left(\frac{\sqrt{1+|\zeta|^2}}{\|z\|_K}\right)^N \sum_{k+l\geq N} (k+l-N+1)\left|a_{k,l}\zeta^k\right| \left(\frac{\|z\|_K}{\sqrt{1+|\zeta|^2}}\right)^{k+l} \\
&\leq \left(\frac{\sqrt{1+|\zeta|^2}}{\|z\|_K}\right)^N \sum_{m\geq N} (m+1) \left(\frac{\|z\|_K}{\sqrt{1+|\zeta|^2}}\right)^m \frac{\|f\|_R}{R^m}\sum_{k=0}^m |\zeta|^k \\
&\leq \|f\|_R \left(\frac{\sqrt{1+|\zeta|^2}}{\|z\|_K}\right)^N \sum_{m\geq N} (m+1)^2 \left(\frac{\|z\|_K|\zeta|}{R\sqrt{1+|\zeta|^2}}\right)^m \\
&\leq \|f\|_R \left(\frac{\sqrt{1+|\zeta|^2}}{\|z\|_K}\right)^N \sum_{m\geq N} (m+1)^2 \left(\frac{\|z\|_K}{R}\right)^m \leq \|f\|_R \left(\frac{\sqrt{1+|\zeta|^2}}{\|z\|_K}\right)^N \sum_{m\geq N} \frac{(m+1)^2}{8^m} \\
&\leq \|f\|_R \left(\frac{\sqrt{1+|\zeta|^2}}{\|z\|_K}\right)^N \sum_{m\geq N} \frac{1}{2^m} \leq 2\|f\|_{R_K} \left(\frac{1+|\zeta|}{2\|z\|_K}\right)^N ,
\end{aligned}
$$

for $R \geq R_K := 8\|z\|_K$. Then for all

$$
R \quad \geq \quad \max\left(R_K, 1+\max_{1\leq j\leq N}|\eta_j|\right) ,
$$

one has

$$
\int_{|\zeta|\geq R}\sum_{k+l\geq N} \frac{(k+l-N+1)\left|a_{k,l}\zeta^k\right|}{\prod_{j=1}^N|\zeta-\eta_j|}\left|\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right|^{k+l-N}\frac{\left|d\overline{\zeta}\wedge d\zeta\right|}{(1+|\zeta|^2)^2} \quad \leq
$$

$$
\leq \quad \frac{2\|f\|_{R_K}}{(2\|z\|_K)^N}\int_0^{2\pi} d\theta \int_R^\infty \frac{(1+r)^N}{\prod_{j=1}^N(r-|\eta_j|)}\frac{2rdr}{(1+r^2)^2} \; < \; +\infty\,.
$$

It follows that one can write, for all $z \in \mathbb{C}^2$,

$$
\begin{aligned}
R_N(f;\eta)(z) \quad &= \quad \sum_{k+l\geq N} a_{k,l}\left[\sum_{p=1}^N\prod_{j\neq p}\left(\frac{z_1-\eta_j z_2}{\eta_p-\eta_j}\right)\eta_p^k\left(\frac{z_2+\overline{\eta_p}\,z_1}{1+|\eta_p|^2}\right)^{k+l-N+1}\right] \\
&= \quad \sum_{k+l\geq N} a_{k,l}z_1^kz_2^l \; - \; \prod_{j=1}^N(z_1-\eta_j z_2) \; \times \\
&\times \quad \frac{1}{2i\pi}\int_{\mathbb{C}}\frac{\sum_{k+l\geq N}(k+l-N+1)a_{k,l}\zeta^k\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{k+l-N+1}}{\prod_{j=1}^N(\zeta-\eta_j)}\frac{d\overline{\zeta}\wedge d\zeta}{(1+|\zeta|^2)^2} ,
\end{aligned}
$$

and the lemma is proved.

$\surd$

Now we can give the proof of Proposition 4.

Proof. Let be $\eta = (\eta_j)_{j\geq 1}$, $\eta' = (\eta_{j_s})_{s\geq 1}$ and $f(z) = \sum_{k,l\geq 0} a_{k,l} z_1^k z_2^l \in \mathcal{O}\left(\mathbb{C}^2\right)$. We want to prove that $R_N(f;\eta\setminus\eta')$ converges to 0 as $N\to\infty$, uniformly on any compact subset. The idea of the proof is to apply the hypothesis of the convergence of $R_N(\cdot;\eta)$ to a holomorphic family that is associate to f and defined below.

First, for all $m\geq 0$, we consider $s_m\geq 1$ such that $j_{s_m}\leq m < j_{s_m+1}$ and we define the following homogeneous sum in $z=(z_1,z_2)$:

$$\prod_{s=1}^{s_m}(z_1-\eta_{j_s}z_2)\sum_{k+l=m-s_m} a_{k,l}z_1^k z_2^l\,.$$

If $0\leq m<j_1$ then we set $s_m := 0$ and the above sum is just $\sum_{k+l=m} a_{k,l}z_1^k z_2^l$. We define the following family of holomorphic functions:

$$f_{\eta',q}(z) \;:=\; \prod_{s=1}^{s_q}(z_1-\eta_{j_s}z_2)\sum_{k+l\geq q-s_q} a_{k,l}z_1^k z_2^l\,, \tag{3.19}$$

for all $q\geq 0$.

Now we claim that the family $\{f_{\eta',q}\}_{q\geq 0}$ is relatively compact in $\mathcal{O}\left(\mathbb{C}^2\right)$. Since $f_{\eta',0}=f$, one can assume that $q\geq 1$. Let $K\subset\mathbb{C}^2$ be any compact subset. For all $R>\|z\|_K$, one has by (3.16), (3.17) and (3.18) from Lemma 8

$$\begin{aligned}
\prod_{s=1}^{s_q}|z_1-\eta_{j_s}z_2|\sum_{k+l\geq q-s_q}\left|a_{k,l}z_1^k z_2^l\right| &\leq \prod_{s=1}^{s_q}\left(\|z\|\sqrt{1+|\eta_{j_s}|^2}\right)\sum_{m\geq q-s_q}\sum_{k+l=m}\frac{\|f\|_R}{R^{k+l}}\|z\|^{k+l}\\
&\leq \left(\|z\|_K\left(1+\|\eta'\|_\infty\right)\right)^{s_q}\|f\|_R\sum_{m\geq q-s_q}\left(\frac{\|z\|_K}{R}\right)^m(m+1)\\
&\leq \|f\|_R\left(\left(1+\|z\|_K\right)\left(1+\|\eta'\|_\infty\right)\right)^{s_q}\times\\
&\quad\times\frac{(q-s_q+1+\|z\|_K/R)(\|z\|_K/R)^{q-s_q}}{(1-\|z\|_K/R)^2}\,.
\end{aligned}$$

Since η' is a subsequence, one has $j_{s_q}\geq s_q$ and by definition of s_q, one has $s_q\leq j_{s_q}\leq q$. On the other hand, one has by hypothesis of η',

$$q-s_q \;=\; \mathrm{card}\left\{\eta_1,\dots,\widehat{\eta_{j_1}},\dots,\eta_j,\dots,\widehat{\eta_{j_{s_q}}},\dots,\eta_q\right\} \;=\; \mathrm{card}\left\{\eta(q)\setminus\eta'\right\} \;\geq\; \frac{q-D}{C}\,.$$

It follows that

$$\begin{aligned}
\sup_{z\in K}|f_{\eta',q}(z)| &\leq \frac{\|f\|_R}{\left(1-\|z\|_K/R\right)^2}\left[\left(1+\|z\|_K\right)\left(1+\|\eta'\|_\infty\right)\right]^q(q-s_q+2)\left(\frac{\|z\|_K}{R}\right)^{(q-D)/C}\\
&\leq \frac{\|f\|_R}{\left(1-\|z\|_K/R\right)^2}\left(\frac{R}{\|z\|_K}\right)^{D/C}(q+2)\left[\frac{(1+\|\eta'\|_\infty)(1+\|z\|_K)\|z\|_K^{1/C}}{R^{1/C}}\right]^q.
\end{aligned}$$

If we fix

$$R \;=\; R_{\eta',K} \;:=\; \max\left\{2\|z\|_K\,,\;\|z\|_K\left[2\left(1+\|\eta'\|_\infty\right)\left(1+\|z\|_K\right)\right]^C\right\}\,,$$

one has for all $q\geq 1$

$$\sup_{z\in K}|f_{\eta',q}(z)| \;\leq\; 4\|f\|_{R_{\eta',K}}\left(R_{\eta',K}/\|z\|_K\right)^{D/C}\frac{q+2}{2^q}\,.$$

We finally get

$$\sup_{q\geq 0}\left\{\sup_{z\in K}|f_{\eta',q}(z)|\right\} \quad\leq\quad \sup_{z\in K}|f(z_1,z_2)| \,+\, 6\left(\frac{R_{\eta',K}}{\|z\|_K}\right)^{D/C}\sup_{|z_1|=|z_2|=R_{\eta',K}}|f(z_1,z_2)|\,.$$

By Lemma 8, one has for all $z\in\mathbb{C}^2$

$$\begin{aligned}R_N\left(f_{\eta',q};\eta\right)(z) \quad &= \\ = \quad &\sum_{k+l\geq N} a_{k,l}\left(f_{\eta',q}\right)z_1^k z_2^l \\ &-\frac{\prod_{j=1}^N\left(z_1-\eta_j z_2\right)}{2i\pi}\int_{\zeta\in\mathbb{C}}\frac{\sum_{k+l\geq N}(k+l-N+1)a_{k,l}\left(f_{\eta',q}\right)\zeta^k\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{k+l-N+1}}{\prod_{j=1}^N\left(\zeta-\eta_j\right)}\frac{d\overline{\zeta}\wedge d\zeta}{(1+|\zeta|^2)^2}\,.\end{aligned}$$

Since the family $\{f_{\eta',q}\}_{q\geq 0}$ is relatively compact in $\mathcal{O}\left(\mathbb{C}^2\right)$, one always has

$$\sup_{q\geq 1}\sup_{z\in K}\left|\sum_{k+l\geq N}a_{k,l}\left(f_{\eta',q}\right)z_1^k z_2^l\right| \quad\xrightarrow[N\to\infty]{} 0\,.$$

On the other hand, since the interpolation formula $E_N(\cdot,\eta)$ is convergent and the family $\{f_{\eta',q}\}_{q\geq 0}$ is uniformly bounded, it follows by Lemma 7 that, on any compact subset K,

$$\sup_{q\geq 1}\sup_{z\in K}\left|R_N\left(f_{\eta',q};\eta\right)(z)\right| \quad\xrightarrow[N\to\infty]{}\quad 0\,.$$

This yields to

$$\sup_{q\geq 0,z\in K}\left|\frac{\prod_{j=1}^N\left(z_1-\eta_j z_2\right)}{2i\pi}\int_{\zeta\in\mathbb{C}}\frac{\sum_{k+l\geq N}(k+l-N+1)a_{k,l}\left(f_{\eta',q}\right)\zeta^k\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{k+l-N+1}d\overline{\zeta}\wedge d\zeta}{\prod_{j=1}^N\left(\zeta-\eta_j\right)\ (1+|\zeta|^2)^2}\right| \xrightarrow[N\to\infty]{} 0\,.$$

In particular, one can choose for $R\geq 1$

$$q \;=\; N \quad\text{ and }\quad K \;=\; C_R \;:=\; C\left(0,(1+\|\eta'\|_\infty)\,R\right)\times C(0,R)\,,$$

then

$$\sup_{|z_1|=(1+\|\eta'\|_\infty)R,\,|z_2|=R}\left|\frac{\prod_{j=1}^N\left(z_1-\eta_j z_2\right)}{2i\pi}\int_{\zeta\in\mathbb{C}}\frac{\sum_{k+l\geq N}(k+l-N+1)a_{k,l}\left(f_{\eta',N}\right)\zeta^k\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{k+l-N+1}d\overline{\zeta}\wedge d\zeta}{\prod_{j=1}^N\left(\zeta-\eta_j\right)\ (1+|\zeta|^2)^2}\right|\xrightarrow[N\to\infty]{} 0\,.$$

On the other hand, by construction of the function

$$f_{q,\eta'}(z) \quad=\quad \prod_{s=1}^{s_q}\left(z_1-\eta_{j_s}z_2\right)\sum_{m\geq q-s_q}\sum_{k+l=m}a_{k,l}z_1^k z_2^l \;=\; \sum_{m\geq q}\sum_{k+l=m}a_{k,l}\left(f_{q,\eta'}\right)z_1^k z_2^l$$

and the uniqueness of its Taylor expansion, one has, for all $m\geq q$ and $z\in\mathbb{C}^2$,

$$\sum_{k+l=m}a_{k,l}\left(f_{q,\eta'}\right)z_1^k z_2^l \quad=\quad \prod_{s=1}^{s_q}\left(z_1-\eta_{j_s}z_2\right)\sum_{k+l=m-s_q}a_{k,l}z_1^k z_2^l\,.$$

It follows that, for all $\zeta \in \mathbb{C}^2$,

$$\sum_{m\geq N}(m-N+1)\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{m-N+1}\sum_{k+l=m}a_{k,l}\left(f_{N,\eta'}\right)\zeta^k \quad =$$

$$\begin{aligned} &= \prod_{s=1}^{s_N}(\zeta-\eta_{j_s})\sum_{m\geq N}(m-N+1)\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{m-N+1}\sum_{k+l=m-s_N}a_{k,l}\zeta^k \\ &= \prod_{s=1}^{s_N}(\zeta-\eta_{j_s})\sum_{m\geq N-s_N}(m-(N-s_N)+1)\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{m-(N-s_N)+1}\sum_{k+l=m}a_{k,l}\zeta^k\,, \end{aligned}$$

then

$$\sup_{|z_1|=(1+\|\eta'\|_\infty)R,\,|z_2|=R}\left|\prod_{s=1}^{s_N}(z_1-\eta_{j_s}z_2)\right|\left|\frac{\prod_{j=1,\,j\neq s_1,\ldots,s_N}^{N}(z_1-\eta_j z_2)}{2i\pi}\right|\times$$

$$\times\left|\int_{\zeta\in\mathbb{C}}\frac{\sum_{k+l\geq N-s_N}(k+l-(N-s_N)+1)a_{k,l}\zeta^k\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{k+l-(N-s_N)+1}d\overline{\zeta}\wedge d\zeta}{\prod_{j=1,\,j\neq s_1,\ldots,s_N}^{N}(\zeta-\eta_j)\ (1+|\zeta|^2)^2}\right|\xrightarrow[N\to\infty]{}0\,.$$

On the other hand, one has for all $R\geq 1$, $|z_1|=(1+\|\eta'\|_\infty)R$ and $|z_2|=R$,

$$\prod_{s=1}^{s_N}|z_1-\eta_{j_s}z_2|\ \geq\ \prod_{s=1}^{s_N}(|z_1|-|\eta_{j_s}z_2|)\ \geq\ \prod_{s=1}^{s_N}[(1+\|\eta'\|_\infty)R-\|\eta'\|_\infty R]\ =\ R^{s_N}\geq 1\,,$$

then one still has

$$\sup_{|z_1|=(1+\|\eta'\|_\infty)R,\,|z_2|=R}\left|\frac{\prod_{j=1,\,j\neq s_1,\ldots,s_N}^{N}(z_1-\eta_j z_2)}{2i\pi}\right|\times$$

$$\times\left|\int_{\zeta\in\mathbb{C}}\frac{\sum_{k+l\geq N-s_N}(k+l-(N-s_N)+1)a_{k,l}\zeta^k\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{k+l-(N-s_N)+1}d\overline{\zeta}\wedge d\zeta}{\prod_{j=1,\,j\neq s_1,\ldots,s_N}^{N}(\zeta-\eta_j)\ (1+|\zeta|^2)^2}\right|\xrightarrow[N\to\infty]{}0\,.$$

Since the compact subset C_R is the Shilov boundary of the closed bidisc, $K_R=\overline{D\left(0,(1+\|\eta'\|_\infty)R\right)}\times\overline{D(0,R)}$ (ie the supremum on K_R of any holomorphic function is still reached on C_R, see for example [2], [5], [6] and [10]), it follows that

$$\sup_{z\in K_R}\left|\frac{\prod_{j=1,\,j\neq s_1,\ldots,s_N}^{N}(z_1-\eta_j z_2)}{2i\pi}\right|\times$$

$$\times\left|\int_{\zeta\in\mathbb{C}}\frac{\sum_{k+l\geq N-s_N}(k+l-(N-s_N)+1)a_{k,l}\zeta^k\left(\frac{z_2+\overline{\zeta}z_1}{1+|\zeta|^2}\right)^{k+l-(N-s_N)+1}d\overline{\zeta}\wedge d\zeta}{\prod_{j=1,\,j\neq s_1,\ldots,s_N}^{N}(\zeta-\eta_j)\ (1+|\zeta|^2)^2}\right|\xrightarrow[N\to\infty]{}0\,.$$

It follows by Lemma 8 that $\{R_{N-s_N}(f;\eta')\}_{N\geq 1}$ converges to 0 uniformly on any compact subset K_R, $R\geq 1$ (and because always so does $\left\{\sum_{k+l\geq N}a_{k,l}z_1^k z_2^l\right\}_{N\geq 1}$). Since any compact subset K can be embedded in some K_R for R large enough, one still has

$$\sup_{z\in K}\left|R_{N-s_N}\left(f;\,\eta\setminus\eta'\right)(z)\right|\quad\xrightarrow[N\to\infty]{}\quad 0\,.$$

On the other hand, from the condition about η', we know that the sequence $(N-s_N)_{N\geq 1}$ is increasing and not bounded. Moreover, by construction of the sequence $(s_N)_{N\geq 1}$, one has in particular $s_N \leq s_{N+1} \leq s_N + 1$. Then $N+1-s_{N+1} \leq N - s_N + 1$, ie the sequence $(N-s_N)_{N\geq 1}$ is surjective. It follows that

$$\lim_{N\to\infty} \sup_{z\in K} |R_N(f;\eta\setminus\eta')(z)| \;=\; \lim_{N'\to\infty} \sup_{z\in K} \left|R_{N'-s_{N'}}(f;\,\eta\setminus\eta')(z)\right| \;=\; 0\,,$$

then by Lemma 7 the interpolation formula $E_N(f;\eta\setminus\eta')$ converges to f uniformly on any compact subset. Since it is true for any function $f\in\mathcal{O}\left(\mathbb{C}^2\right)$, the proposition is proved.

$\surd$

One can deduce as a consequence a special case where the subsequence η' is a finite subset of $\{\eta_j\}_{j\geq 1}$. This result is usefull in order to prove Lemma 6 from the previous subsection (and then to complete the proof of Proposition 1 given in Introduction).

Corollary 6. *Let be a sequence $\eta=(\eta_j)_{j\geq 1}$ whose interpolation formula $E_N(\cdot;\eta)$ converges. For $n\geq 1$, let be $\eta_{j_1},\ldots,\eta_{j_n}$ and consider the new sequence $\widetilde{\eta}$ where*

$$\{\widetilde{\eta}_j\}_{j\geq 1} \;=\; \{\eta_j\}_{j\geq 1}\setminus\{\eta_{j_1},\ldots,\eta_{j_n}\}\,,$$

and such that the order is the same: if $\widetilde{\eta}_s=\eta_{j_s}$ and $\widetilde{\eta}_t=\eta_{j_t}$ with $j_s<j_t$, then $s<t$.

Then the interpolation formula $E_N(\cdot;\widetilde{\eta})$ converges too.

Proof. The proof is exactly the same as the one of Proposition 4 since its conditions are immediatly satisfied. The only difference is that the family $\{f_{\eta',q}\}_q$ is finite then automatically relatively compact.

In addition, it is sufficient to assume the simple convergence of $E_N(\cdot;\eta)$ function by function, and not uniformly on any compact subset of $\mathcal{O}\left(\mathbb{C}^2\right)$ (but as said in Remark 3.1, this is not a really stronger condition).

$\surd$

3.4. **Proof of Theorems 3 and 4.** In this section we give the proof of Theorems 3 and 4. We first begin with the following result that will be usefull. It also gives an answer to the following problem: we know that, if $\{\kappa_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves, then so will be any subset (finite or infinite), in particular if it is a convergent subsequence. Conversely, if $\{\eta_j\}_{j\geq 1}$ is not locally interpolable by real-analytic curves, what about any subset?

In general, it is false (see for example Proposition 1 where $(\kappa_j)_{j\geq 1}$ is locally interpolable by real-analytic curves but the whole sequence $(\eta_j)_{j\geq 1}$ is not). Nevertheless, one can extract subsequences that will still not be locally interpolable by real-analytic curves. But what about if we also want the subsequence to be convergent? The answer is affirmative.

Lemma 9. *Let $\{\eta_j\}_{j\geq 1}$ that is not locally interpolable by real-analytic curves. Then there is a subset $\{\eta_{j_k}\}_{k\geq 1}$ of $\{\eta_j\}_{j\geq 1}$ that satisfies the following conditions:*

- *the sequence $(\eta_{j_k})_{k\geq 1}$ is convergent (in $\overline{\mathbb{C}}$);*
- *the set $\{\eta_{j_k}\}_{k\geq 1}$ is not locally interpolable by real-analytic curves.*

Proof. Since the set η is not locally interpolable by real-analytic curves, there is $\zeta_0 \in \overline{\{\eta_j\}_{j\geq 1}}$ without any neighborhood $V \in \mathcal{V}(\zeta_0)$ and holomorphic function $g \in \mathcal{O}(V_{\zeta_0})$ that can interpolate the conjugate function on $\{\eta_j\}_{j\geq 1} \bigcap V$, ie

$$\forall V \in \mathcal{V}(\zeta_0), \forall g \in \mathcal{O}(V), \exists \eta_j \in V, \; g(\eta_j) \neq \overline{\eta_j}. \tag{3.20}$$

We first assume that $\zeta_0 \neq \infty$. On the other hand, ζ_0 cannot be isolated in $\overline{\{\eta_j\}_{j\geq 1}}$ (otherwise, by taking V_{ζ_0} such that $\{\eta_j\}_{j\geq 1} \bigcap V_{\zeta_0} = \{\zeta_0\}$ and $g_{\zeta_0}(z) \equiv \overline{\zeta_0}$, we would get a contradiction). Then let be some subset $\{\eta_{j_k}\}_{j\geq 1}$ that is a sequence that converges to ζ_0 and such that $\eta_{j_k} \neq \zeta_0$, $\forall k \geq 1$. We set $S_0 := (\eta_{j_k})_{k\geq 1}$.

Now we consider $V_1 := D(\zeta_0, 1)$ and we set $S_1 := S_0 \bigcap V_1$. By construction, S_1 is still a sequence that converges to ζ_0. If S_1 is not locally interpolable by real-analytic curves, the lemma is proved. Otherwise, there are $\widetilde{V}_1 \in \mathcal{V}(\zeta_0)$ and $g_1 \in \mathcal{O}\left(\widetilde{V}_1\right)$ such that, $\forall \eta_j \in S_1$, $\overline{\eta_j} = g_1(\eta_j)$. By reducing $\widetilde{V}_1$ if necessary, we can assume $\widetilde{V}_1$ connected and $\widetilde{V}_1 \subset V_1$. Since ζ_0 satisfies the condition (3.20), if $V_2 = D(\zeta_0, 1/2) \bigcap \widetilde{V}_1$, $g_1|_{V_2}$ cannot interpolate the conjugate function on $V_2 \bigcap \{\eta_j\}_{j\geq 1}$. Then there is $\eta_{s_1} \in V_2 \bigcap \{\eta_j\}_{j\geq 1}$ such that $g_1(\eta_{s_1}) \neq \overline{\eta_{s_1}}$. We set $S_2 := \{\eta_{s_1}\} \bigcup S_1$ and S_2 is still a sequence that converges to ζ_0.

Now if S_2 is not locally interpolable by real-analytic curves, the lemma is proved. Otherwise we repeat the process by considering $\widetilde{V}_2 \subset V_2$ (with $\widetilde{V}_2$ connected), $g_2 \in \mathcal{O}\left(\widetilde{V}_2\right)$, $V_3 = D(\zeta_0, 1/4) \bigcap \widetilde{V}_2$, $\eta_{s_2} \in V_3 \bigcap \{\eta_j\}_{j\geq 1}$ with $g_2(\eta_{s_2}) \neq \overline{\eta_{s_2}}$, and $S_3 := \{\eta_{s_2}\} \bigcup S_2$. By induction on $r \geq 1$, one can construct $S_r = S_1 \bigcup \{\eta_{s_1}, \ldots, \eta_{s_r}\}$ (that is still a sequence that converges to ζ_0) and $V_r \subset D\left(\zeta_0, 1/2^{r-1}\right)$. As long as the sequence S_r is locally interpolable by real-analytic curves, one can consider $\widetilde{V}_r \in \mathcal{V}(\zeta_0)$, with $\widetilde{V}_r$ connected, and $g_r \in \mathcal{O}\left(\widetilde{V}_r\right)$ such that, $\forall \eta_j \in S_r \bigcap \widetilde{V}_r$, $g_r(\eta_j) = \overline{\eta_j}$. We set $V_{r+1} := D(\zeta_0, 1/2^r) \bigcap \widetilde{V}_r$ and, since by hypothesis g_r cannot interpolate the conjugate function on $V_{r+1} \bigcap \{\eta_j\}_{j\geq 1}$, there is $\eta_{s_{r+1}} \in V_{r+1}$ such that $g_r\left(\eta_{s_{r+1}}\right) \neq \overline{\eta_{s_{r+1}}}$. Then we set $S_{r+1} := \left\{\eta_{s_{r+1}}\right\} \bigcup S_r$ (that still converges to ζ_0).

Now exactly one of two things can happen: either there is $r \geq 1$ such that S_r is not locally interpolable by real-analytic curves, then the proof is achieved; otherwise we can construct by induction on $r \geq 1$ the sequence S_r (that converges to ζ_0), $\widetilde{V}_r \in \mathcal{V}(\zeta_0)$ (connected), $g_r \in \mathcal{O}\left(\widetilde{V}_r\right)$ such that $g_r(\eta_j) = \overline{\eta_j}$, $\forall \eta_j \in S_r \bigcap \widetilde{V}_r$, and $S_{r+1} = \left\{\eta_{s_{r+1}}\right\} \bigcup S_r$ with $g_r\left(\eta_{s_{r+1}}\right) \neq \overline{\eta_{s_{r+1}}}$.

We consider $S_\infty := \bigcup_{r\geq 1} S_r$ the limit of the sequences S_r. S_∞ is still a sequence that converges to ζ_0 (as the union of S_0 and the convergent sequence $\{\eta_{s_r}\}_{r\geq 1}$ since $\eta_{s_r} \in D\left(\zeta_0, 1/2^{r-1}\right)$, $\forall r \geq 1$). We claim that S_∞ cannot be locally interpolable by real-analytic curves. Otherwise, there would be $V_\infty \in \mathcal{V}(\zeta_0)$ and $g_\infty \in \mathcal{O}(V_\infty)$ such that, $\forall \eta_j \in V_\infty \bigcap S_\infty$, $g_\infty(\eta_j) = \overline{\eta_j}$. Let also be $r_0 \geq 1$ such that $\widetilde{V}_{r_0} \subset V_\infty$ (such an r_0 exists since $\widetilde{V}_r \subset D\left(\zeta_0, 1/2^{r-1}\right)$, $\forall r \geq 1$, and V_∞ is a neighborhood of ζ_0). In particular (since $S_{r_0} \subset S_\infty$), $\forall \eta_j \in S_{r_0} \bigcap \widetilde{V}_{r_0}$, one has $g_\infty(\eta_j) = \overline{\eta_j}$. On the other hand, one still has by construction $g_{r_0}(\eta_j) = \overline{\eta_j}$, $\forall \eta_j \in S_{r_0} \bigcap \widetilde{V}_{r_0}$. It follows that g_{r_0} and $g_\infty|_{\widetilde{V}_r}$ are both holomorphic functions on the domain $\widetilde{V}_{r_0}$ that coincide on the subset $S_{r_0} \bigcap \widetilde{V}_{r_0}$ that is infinite with limit point $\zeta_0 \in \widetilde{V}_{r_0}$, then $g_\infty|_{\widetilde{V}_{r_0}} = g_{r_0}$. But in this case, since $\eta_{s_{r_0+1}} \in V_{r_0+1} \subset \widetilde{V}_{r_0} \subset V_\infty$ and $\eta_{s_{r_0+1}} \in S_{r_0+1} \subset S_\infty$, this

yields to

$$\overline{\eta_{s_{r_0+1}}} \neq g_{r_0}\left(\eta_{s_{r_0+1}}\right) = g_\infty\left(\eta_{s_{r_0+1}}\right) = \overline{\eta_{s_{r_0+1}}},$$

and that is impossible. It follows that the set S_∞ satisfies the required conditions and the lemma is proved in the case $\zeta_0 \neq \infty$.

Now if we assume that $\zeta_0 = \infty$, then by removing 0 from $\{\eta_j\}_{j\geq 1}$ if necessary, one can assume that $\eta_j \neq 0, \forall j \geq 1$. Indeed, ∞ will still be a limit point of $\{\eta_j\}_{j\geq 1} \setminus \{0\}$; on the other hand, $\forall V \in \mathcal{V}(\infty)$, with $0 \notin V$, and $g \in \mathcal{O}(V)$, there is $\eta_j \in V$ such that $g(\eta_j) \neq \overline{\eta_j}$ (with $\eta_j \neq 0$), ie ∞ still satisfies the condition (3.20) with the set $\{\eta_j\}_{j\geq 1} \setminus \{0\}$.

Now consider the set $1/\eta := \{1/\eta_j\}_{j\geq 1}$ (that is well-defined). We claim that (the limit point) 0 satisfies the condition (3.20) with $1/\eta$. Otherwise, there would be $V_0 \in \mathcal{V}(0)$ and $g_0 \in \mathcal{O}(V_0)$ such that, for all $1/\eta_j \in V_0$, one would have $g_0(1/\eta_j) = \overline{1/\eta_j}$. In particular, by choosing a subsequence $(\eta_{j_k})_{k\geq 1}$ that converges to ∞ (since it is a limit point), one would have $g_0(0) = 0$. The function h defined on a neighborhood $V_\infty \in \mathcal{V}(\infty)$ by $h(\zeta) := 1/[g_0(1/\zeta)]$, would be holomorphic from V_∞ to $\overline{\mathbb{C}}$ and satisfy:

$$\forall\, \eta_j \in V_\infty,\ h(\eta_j) = \frac{1}{g_0(\eta_j)} = \frac{1}{\overline{1/\eta_j}} = \overline{\eta_j},$$

ie ∞ would not satisfy any more the condition (3.20) with the set η, and that is impossible.

Since $0 \neq \infty$, it follows that we are in the first case and one can apply the lemma: there is a subsequence $(1/\eta_{j_k}) \subset (1/\eta_j)_{j\geq 1}$ that converges to 0 and is not locally interpolable by real-analytic curves. Then the sequence $(\eta_{j_k})_{k\geq 1}$ converges to ∞ and is not either locally interpolable by real-analytic curves (otherwise so would be $(1/\eta_{j_k})_{k\geq 1}$ by applying the above argument).

$\surd$

Before giving the proof of Theorem 3, we remind the following result whose proof is given in [12] (Lemma 12).

Lemma 10. *Let $(\eta_j)_{j\geq 1}$ be any sequence. For any $u \notin \{\eta_j\}_{j\geq 1} \cup \{\infty\}$, consider the following homographic application*

$$h_u : \overline{\mathbb{C}} \to \overline{\mathbb{C}}, \qquad \zeta \mapsto \frac{1 + \overline{u}\,\zeta}{\zeta - u}, \tag{3.21}$$

and the new sequence $\theta = (\theta_j)_{j\geq 1} := (h_u(\eta_j))_{j\geq 1}$.

Then the formula $R_N(f;\eta)$ converges to 0, uniformly on any compact subset and for every function $f \in \mathcal{O}(\mathbb{C}^2)$, if and only if so does $R_N(f;\theta)$ (constructed with the associate θ_j's).

One can deduce the following result.

Corollary 7. *η being any sequence, h_u any homographic application like (3.21) and $\theta := h_u(\eta)$, the formula $E_N(f;\eta)$ converges to f, uniformly on any compact subset and for every function $f \in \mathcal{O}(\mathbb{C}^2)$, if and only if so does $E_N(f;\theta)$.*

Proof. By Lemma 7, the formula $E_N(\cdot;\eta)$ converges if and only if $R_N(\cdot;\eta)$ converges to 0 (uniformly on any compact subset). By Lemma 10, this is true if and only if so does $R_N\left(\cdot;h_u(\eta)\right)$, for any homographic application h_u of the type (3.21). Finally, by applying Lemma 7 again, it is true if and only if $E_N\left(\cdot;h_u(\eta)\right)$ is convergent.

$\surd$

Now we can give the proof of Theorem 3.

Proof. The first sense of the equivalence is obvious since the property of local interpolation by real-analytic curves is a condition about sets, then it does not depend on the numeration of the set $\eta = \{\eta_j\}_{j\geq 1}$.

Now assume that the sequence $\eta = (\eta_j)_{j\geq 1}$ is such that, for all $\sigma \in \mathfrak{S}_{\mathbb{N}}$, the sequence $\sigma(\eta) := \left(\eta_{\sigma(j)}\right)_{j\geq 1}$ makes converge its associate interpolation formula $E(\cdot;\sigma(\eta))$.

First, let assume that η is bounded, ie $\|\eta\|_\infty := \sup_{j\geq 1}|\eta_j| < +\infty$ and assume that η is not locally interpolable by real-analytic curves. Then by Lemma 9, there is a subsequence $(\eta_{j_k})_{k\geq 1}$ that converges to $\zeta_0 \in \overline{\{\eta_j\}_{j\geq 1}}$ and that is not locally interpolable by real-analytic curves. On the other hand, the sequence $(\eta_j)_{j\geq 1} \setminus (\eta_{j_k})_{k\geq 1}$ can be rewritten as $(\eta_{s_k})_{k\geq 1}$. We define $\sigma \in \mathfrak{S}_{\mathbb{N}}$ as follows:

$$\sigma(k) \quad := \quad \begin{cases} j_{k/2} \text{ if } k \text{ is even}\,, \\ s_{(k-1)/2} \text{ if } k \text{ is odd}\,. \end{cases}$$

By construction σ is a permutation of $\mathbb{N}\setminus\{0\}$ then by hypothesis the sequence $\left(\eta_{\sigma(k)}\right)_{k\geq 1}$ still makes converge its associate interpolation formula $E_N(\cdot;\sigma(\eta))$. If we consider the subsequence $\eta' := \left(\eta_{\sigma(2k+1)}\right)_{k\geq 1}$, then one has $\left(\eta_{\sigma(2k+1)}\right)_{k\geq 1} = (\eta_{s_k})_{k\geq 1}$ and for all even (resp. odd) $N \geq 1$, one has

$$\begin{aligned} \operatorname{card}\{\sigma(\eta)(N)\setminus\eta'\} &= \operatorname{card}\left\{\eta_{\sigma(2)},\eta_{\sigma(4)},\ldots,\eta_{\sigma(N)}\right\} = \operatorname{card}\left\{\eta_{j_1},\eta_{j_2},\ldots,\eta_{j_{N/2}}\right\} \\ &= \frac{N}{2} \geq \frac{N-1}{2} \end{aligned}$$

(resp. $\operatorname{card}\{\eta_2,\eta_4,\ldots,\eta_{N-1}\} = (N-1)/2$). On the other hand, η' is bounded (as a subsequence of the bounded sequence $(\eta_j)_{j\geq 1}$). It follows by Proposition 4 that the new sequence $\eta\setminus\eta' = (\eta_{j_k})_{k\geq 1}$ still makes converge its associate interpolation formula.

But we also know that $(\eta_{j_k})_{k\geq 1}$ is a convergent sequence that is not locally interpolable by real-analytic curves, then by Proposition 2 it cannot make converge its associate interpolation formula. This is impossible and the conclusion is that the set $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves.

Now let assume that the sequence η is not dense, ie $\overline{\{\eta_j\}_{j\geq 1}} \neq \mathbb{C}$ and let be $u \notin \overline{\{\eta_j\}_{j\geq 1}}$ (one can assume that u is finite, otherwise η is bounded). Consider the homographic application

$$\begin{aligned} h\,:\,\overline{\mathbb{C}} &\to \overline{\mathbb{C}} \\ \zeta &\mapsto \frac{1+\overline{u}\,\zeta}{\zeta-u}\,. \end{aligned}$$

It follows that the sequence $\theta = (\theta_j)_{j\geq 1} := h(\eta) := (h(\eta_j))_{j\geq 1}$ is bounded.

On the other hand, since the interpolation formula $E_N(\cdot;\sigma(\eta))$ converges for all $\sigma \in \mathfrak{S}_{\mathbb{N}}$, then by Corollary 7 so does $E_N(\cdot;\sigma(\theta))$. Thus one can apply the first case above of the proof and deduce that the set $\{\theta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves. Finally, the appplication h being homographic, in particular it is biholomorphic. It follows that the set $\{\eta_j\}_{j\geq 1} = \left\{h^{(-1)}(\theta_j)\right\}_{j\geq 1}$ is locally interpolable by real-analytic curves too.

The last case with which we have to deal is the one when the sequence η is dense, ie $\overline{\{\eta_j\}_{j\geq 1}} = \mathbb{C}$. In particular, its topological closure has nonempty interior. It follows by Lemma 1 that it cannot be locally interpolable by real-analytic curves. In order to complete the equivalence in this case, we want to prove that there is $\sigma \in \mathfrak{S}_{\mathbb{N}}$ such that $E_N(\cdot;\sigma(\eta))$ does not converge.

Assume that it is not the case, ie for all $\sigma \in \mathfrak{S}_{\mathbb{N}}$ and all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, the interpolation formula $E_N(f;\sigma(\eta))$ converges to f. Without loss of generality, one can assume that, for all $k \geq 1$, one has $|\eta_{2k}| \leq 1$ and $|\eta_{2k-1}| > 1$. Indeed, if it is not the case, let consider $\eta' := \{|\eta_j| \leq 1\}$, that can be rewritten as $(\eta_{j_k})_{k\geq 1}$. The other set $\eta \setminus \eta'$ can be rewritten as $(\eta_{s_k})_{k\geq 1}$. Both subsequences are well-defined since η is assumed to be dense. We analogously define the permutation σ_a by

$$\sigma_a(k) \quad = \quad \begin{cases} j_{k/2} & \text{if } k \text{ is even,} \\ s_{(k+1)/2} & \text{if } k \text{ is odd.} \end{cases}$$

Then by hypothesis, for all permutation $\sigma \in \mathfrak{S}_{\mathbb{N}}$, the composition $\sigma\sigma_a$ is still a permutation of $\mathbb{N}\backslash\{0\}$ then the interpolation formula $E_N(\cdot;\sigma(\sigma_a(\eta))) = E_N(\cdot;(\sigma\sigma_a)(\eta))$ still converges, ie the new sequence $\sigma_a(\eta)$ satisfies the same conditions as η does.

Now we claim that $\eta \setminus \eta'$ also satisfies the same conditions as η does, ie for all $\sigma \in \mathfrak{S}_{\mathbb{N}}$ and all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, the interpolation formula $E_N(f;\sigma(\eta\setminus\eta'))$ converges. Indeed, let σ be any permutation of $\mathbb{N}$. Such a permutation σ will act as $\sigma(\eta\setminus\eta') = \left(\eta_{s_{\sigma(k)}}\right)_{k\geq 1} = \left(\eta_{2\sigma(k)-1}\right)_{k\geq 1}$. We canonically extend σ to $\widetilde{\sigma}$ for the whole sequence η by fixing η', ie by setting:

$$\widetilde{\sigma}(j) \quad := \quad \begin{cases} j & \text{if } j \text{ is even,} \\ 2\sigma((j+1)/2)-1 & \text{if } j \text{ is odd.} \end{cases}$$

$\widetilde{\sigma}$ is well-defined and is a permutation of $\mathbb{N}$ that fixes all the even j's, ie $\widetilde{\sigma}(\eta') = (\eta_{2k})_{k\geq 1} = \eta'$. On the other hand, for all $N \geq 1$, one has

$$\operatorname{card}\{[\widetilde{\sigma}(\eta)](N)\setminus\widetilde{\sigma}(\eta')\} \;=\; \operatorname{card}\{[\widetilde{\sigma}(\eta)](N)\setminus\eta'\} \;=\; \operatorname{card}\left\{\eta_{2\sigma(k)-1},\, 2k-1 \leq N\right\} \;\geq\; \frac{N}{2}$$

(since $\widetilde{\sigma}$ globally fixes the set of the odd j's). Since $\widetilde{\sigma}(\eta') = \eta'$ is still bounded, it follows by Proposition 4 that the new sequence

$$\sigma(\eta\setminus\eta') \;=\; \left(\eta_{2\sigma(k)-1}\right)_{k\geq 1} \;=\; \left(\eta_{\widetilde{\sigma}(2k-1)}\right)_{k\geq 1} \;=\; \widetilde{\sigma}(\eta)\setminus\widetilde{\sigma}(\eta')$$

still makes converge its associate interpolation formula.

The consequence is the following: the new sequence $\eta\setminus\eta'$ is not dense and makes converge its associate interpolation formula under the action of any permutation $\sigma \in \mathfrak{S}_{\mathbb{N}}$. One can apply the second case above and deduce that $\eta\setminus\eta'$ is locally interpolable by real-analytic curves. In particular, it follows by Lemma 1 that its topological closure has empty interior. But this is impossible since $\overline{\eta\setminus\eta'} =$

$\{z \in \mathbb{C},\, |z| \geq 1\}$. The conclusion is that a dense sequence cannot keep making converge its associate interpolation formula under the action of any permutation. This achieves the equivalence and the proof of the theorem.

$\surd$

Now we give the proof of Theorem 4.

Proof. First, if $(\eta_j)_{j\geq 1}$ is locally interpolable by real-analytic curves, then so is any subsequence $(\eta_{j_k})_{k\geq 1}$. The assertion follows by Theorem 2.

Conversely, assume that $(\eta_j)_{j\geq 1}$ is not locally interpolable by real-analytic curves. It follows by Lemma 9 that there is a subsequence $\eta' = (\eta_{j_k})_{k\geq 1}$ that is convergent (in $\overline{\mathbb{C}}$) and that is not locally interpolable by real-analytic curves. If we prove that η' does not make converge its associate interpolation formula, the proof of the theorem will be achieved.

Let be $\eta_\infty = \lim_{k\to\infty} \eta_{j_k}$. If we assume that η_∞ is finite, it will follow by Proposition 2 that the subsequence η' cannot make converge its associate interpolation formula. Then one can assume that $\eta_\infty = \infty$. By a taking sub-subsequence of η' if necessary, one can assume that $\eta_{j_k} \neq 0$, $\forall\, k \geq 1$ (and this sub-subsequence will still satisfy the same conditions as η' does). In this case, Lemma 9 means that the sequence $1/\eta' := (1/\eta_{j_k})_{k\geq 1}$ converges to 0 and is not locally interpolable by real-analytic curves. Since $1/\eta'$ is bounded, one can apply Proposition 2 to deduce that the interpolation formula $E_N(\cdot; 1/\eta')$ does not converge. It follows by Corollary 7 with the choice of the homographic application $h_0(\zeta) = 1/\zeta$, that neither can $E_N(\cdot; h_0(1/\eta'))$ converge, ie $E_N(\cdot; \eta')$ does not converge and the proof of the theorem is achieved.

$\surd$

4. On the case of a dense family

In this part we deal with the case of a dense sequence

$$\overline{\{\eta_j\}_{j\geq 1}} \;=\; \mathbb{C}\,,$$

and we give the proof of Theorem 5. We already know from Theorem 3 that, since its topological closure has nonempty interior, then there is a permutation $\sigma_d \in \mathfrak{S}_{\mathbb{N}}$ such that the interpolation formula $E_N(\cdot; \sigma_d(\eta))$ will not converge. Thus the proof will be showing that there is also $\sigma_c \in \mathfrak{S}_{\mathbb{N}}$ such that $E_N(\cdot; \sigma_c(\eta))$ converges.

The idea of the proof is the following: we want to get a numeration of η that is of the same kind as in Proposition 3, ie by taking advantage of the ε-entropy of a 2-dimensional open set so that the points $\eta_{\sigma_c(j)}$ will certainly be close from each other but not too fast; on the other hand, the numeration will be defined so that the point ∞ will not be approached too fast by the new sequence $\sigma_c(\eta)$. In fact, we want to get a new numeration of η that is uniform on the Riemann sphere $\overline{\mathbb{C}}$.

4.1. **The construction of $\sigma_c(\eta)$:** We set $C_1 := [-1,1]+i[-1,1]$ and for all $r \geq 2$, we define the following set

$$C_r \quad := \quad \{[-r,r]+i[-r,r]\} \setminus \{[-(r-1),r-1]+i[-(r-1),r-1]\} \ . \tag{4.1}$$

One has the following partition

$$\mathbb{C} \quad = \quad \bigcup_{r\geq 1} C_r \, .$$

For all $r \geq 1$, we consider each subsequence

$$\eta^{(r)} \;=\; \left(\eta_{\Phi_r(k)}\right)_{k\geq 1} \quad := \quad \eta \cap C_r \, ,$$

with its natural numeration from the one of η. Then each Φ_r is injective and form a partition of $\mathbb{N} \setminus \{0\}$, ie

$$\mathbb{N}\setminus\{0\} \quad = \quad \bigcup_{r\geq 1} \{\Phi_r(k),\, k\geq 1\} \ .$$

In addition, since one has $\eta = \bigcup_{r\geq 1} \eta^{(r)}$, it follows that each $\eta^{(r)}$ is a dense sequence in C_r.

In order to construct σ_c, first we want to construct each $\sigma_c^{(r)}$ that is associate to $\eta^{(r)}$. The idea will be using for each $\eta^{(r)}$ the same argument as in Proposition 3.

4.1.1. *The construction of each $\sigma^{(r)}$:* Since $r \geq 1$ is fixed, in order to avoid heavy notations, we will just write σ and $\eta = (\eta_j)_{j\geq 1}$ instead of $\sigma_c^{(r)}$ and $\eta^{(r)} = \left(\eta_{\Phi_r(k)}\right)_{k\geq 1}$.

Let be $\eta_1 \in C_r$ and $k_1 \geq 3$ the smallest integer such that

$$|\eta_1| \quad \geq \quad r-1+\frac{1}{2^{k_1}} \, .$$

We complete η_1 by an $1/2^{k_1}$-net of

$$C_r \setminus \left\{[-r+1-1/2^{k_1}, r-1+1/2^{k_1}] + i[-r+1-1/2^{k_1}, r-1+1/2^{k_1}]\right\} \, ,$$

whose cardinal will be of order

$$\begin{aligned} 4r^2 2^{2k_1} - 4\left(r-1+\frac{1}{2^{k_1}}\right)^2 2^{2k_1} \quad &\sim \quad 4 2^{2k_1}\left[r^2 - r^2 + 2r - \frac{2r}{2^{k_1}} + \frac{2}{2^{k_1}} - \frac{1}{2^{2k_1}}\right] \\ &\sim \quad 8r 2^{2k_1}\left(1-\frac{1}{2^{k_1}}\right) \;\sim\; \gamma_{r,k_1} 2^{2k_1} \, , \end{aligned}$$

with

$$\gamma_{r,k} \quad := \quad 8r\left(1-\frac{1}{2^k}\right) \, . \tag{4.2}$$

If $r = 1$, it will be just $4 2^{2k_1}$, ie $\gamma_{1,k} := 4$ (we remind that the cardinal of an ε-net to cover $[-r,r]+i[-r,r]$ is of order $(2r/\varepsilon)^2$).

Now we use the density of η to approach this $1/2^{k_1}$-net by a subset $\mathcal{N}_{k_1}$ so that its cardinal will have the same order. The mutual distance of these $\eta_j \in \mathcal{N}_{k_1}$ will necessary be at least of order $\gamma_{r,k_1} 2^{2k_1}$. Then as in Proposition 3, we use the fact that

$$\mathcal{N}_{k_1} \quad = \quad \bigcup_{k=1}^{k_1} \mathcal{N}_k$$

where $\mathcal{N}_k \subset \mathcal{N}_{k_1}$ is an $1/2^k$-net whose cardinal will be of order $\gamma_{r,k} 2^{2k}$. This allows us to define σ as follows: we consider any numeration of $\mathcal{N}_1$, then any numeration of $\mathcal{N}_2 \setminus \mathcal{N}_1$ and so on until reaching $\mathcal{N}_{k_1}$. This defines σ for the first $\gamma_{r,k} 2^{2k_1}$ points. In particular, η_1 is reached.

Now let $n_1 \geq 1$ be the biggest integer such that $\eta_j \in \mathcal{N}_{k_1}$, $\forall\, j = 1, \ldots, n_1$. We analogously define $k_2 \geq k_1 + 1$ the smallest integer such that

$$\min \left\{ |\eta_{n_1+1}| - (r-1)\,,\ \min_{\eta_j \in \mathcal{N}_{k_1}} |\eta_{n_1+1} - \eta_j| \right\} \quad \geq \quad \frac{1}{2^{k_2}}\,.$$

As above, by using the density of $\{\eta_j\}_{j\geq 1} \setminus \mathcal{N}_{k_1}$, we analogously complete $\mathcal{N}_{k_1} \cup \{\eta_{n_1+1}\}$ on an perturbed $1/2^{k_2}$-net that we call $\mathcal{N}_{k_2}$ and whose cardinal is of order $\gamma_{r,k_2} 2^{2k_2}$. In addition, one can split $\mathcal{N}_{k_2}$ as a union of $\mathcal{N}_k$, from 1 to k_2, whose cardinal will be of order $\gamma_{r,k} 2^{2k}$. This allows us to define σ for $\mathcal{N}_{k_2} \setminus \mathcal{N}_{k_1}$ as above. Moreover, since $\eta_{n_1+1} \in \mathcal{N}_{k_2}$, then in particular $\eta_2 \in \mathcal{N}_{k_2}$.

More generally, let assume having constructed $\mathcal{N}_{k_1}, \ldots, \mathcal{N}_{k_q}$ such that each $\mathcal{N}_k$, $1 \leq k \leq k_q$, is a perturbed $1/2^k$-net of $\{\eta_j\}_{j\geq 1}$ whose cardinal is of order $\gamma_{r,k} 2^{2k}$ and such that $\eta_j \in \mathcal{N}_{k_j}$, $\forall\, j = 1, \ldots, q$. Then we consider $n_q \geq q$ the biggest integer such that $\eta_j \in \mathcal{N}_{k_q}$, $\forall\, j = 1, \ldots, n_q$, and k_{q+1} the smallest integer such that

$$\min \left\{ \left|\eta_{n_q+1}\right| - (r-1)\,,\ \min_{\eta_j \in \mathcal{N}_{k_q}} \left|\eta_{n_q+1} - \eta_j\right| \right\} \quad \geq \quad \frac{1}{2^{k_{q+1}}}\,.$$

Since $\mathcal{N}_{k_q}$ is finite, one can use the density of $\{\eta_j\}_{j\geq 1} \setminus \mathcal{N}_{k_q}$ in C_r to complete $\mathcal{N}_{k_q} \cup \left\{\eta_{n_q+1}\right\}$ on a perturbed $1/2^{k_{q+1}}$-net of $\{\eta_j\}_{j\geq 1}$ that we call $\mathcal{N}_{k_{q+1}}$ and whose cardinal is of order $\gamma_{r,k_{q+1}} 2^{2k_{q+1}}$. On the other hand, one can write

$$\mathcal{N}_{k_{q+1}} \quad = \quad \bigcup_{k=1}^{k_{q+1}} \mathcal{N}_k\,,$$

where $\mathcal{N}_k$ is a perturbed $1/2^k$-net whose cardinal is of order $\gamma_{r,k} 2^{2k}$. In addition, since $\eta_{n_q+1} \in \mathcal{N}_{k_{q+1}}$, then in particular $\eta_{q+1} \in \mathcal{N}_{k_{q+1}}$. We finally extend σ on $\mathcal{N}_{k_{q+1}}$ by successively defining it on each $\mathcal{N}_{k+1} \setminus \mathcal{N}_k$, $k = k_q, \ldots, k_{q+1} - 1$. Notice that, as in Proposition 3, the numeration on each $\mathcal{N}_{k+1} \setminus \mathcal{N}_k$ does not matter since the most important is to fill each of them before going to the next one.

This shows that the application σ is well-defined, injective by construction and surjective since, for all $j \geq 1$, one has $\eta_j \in \mathcal{N}_{k_j}$.

4.1.2. *The construction of σ_c on the whole sequence η:* Now that we have defined $\sigma_c^{(r)}$ on each C_r, we want construct the global application σ_c such that, for all $r \geq 1$, one has

$$\#\left\{(\sigma_c \eta)\,(N) \cap C_r\right\} \;:=\; \mathrm{card}\left\{j,\, 1 \leq j \leq N,\, \eta_{\sigma_c(j)} \in C_r\right\} \quad \sim \quad \frac{N}{2^r}\,. \tag{4.3}$$

We begin with choosing $\eta_{\sigma_c(1)}, \eta_{\sigma_c 2)} \in C_1$, then $\eta_{\sigma_c(1)} := \eta_{\sigma_c^{(1)}(1)}$ and $\eta_{\sigma_c(2)} := \eta_{\sigma_c^{(1)}(2)}$ (ie $\sigma_c(1) := \sigma_c^{(1)}(1)$ and $\sigma_c(2) := \sigma_c^{(1)}(2)$). Next, we set $\sigma_c(3) := \sigma_c^{(2)}(1)$ so that $\eta_{\sigma_c(3)} \in C_2$, go back to C_1 and so on.

More generally, let assume having defined σ_c on $\{1,\ldots,N\}$ such that for all $r\geq 1$, one has:

$$\frac{\#\{(\sigma_c\eta)(N)\cap C_r\}}{2}-1 \;\leq\; \#\{(\sigma_c\eta)(N)\cap C_{r+1}\} \;\leq\; \frac{\#\{(\sigma_c\eta)(N)\cap C_r\}}{2}\,, \tag{4.4}$$

and

$$\{(\sigma_c\eta)(N)\cap C_r\} \;=\; \left\{\eta_{\sigma_c^{(r)}(j)},\, j=1,\ldots,\#\{(\sigma_c\eta)(N)\cap C_r\}\right\} \tag{4.5}$$

(this last relation just means that, in each C_r, we naturally take the first $\eta_{\sigma_c^{(r)}(j)}$'s).

We want to know which one will be the candidate for $\eta_{\sigma_c(N+1)}$. Let consider r_N the biggest $r\geq 1$ so that $(\sigma_c\eta)(N)\cap C_r$ is not empty. By (4.4), one has

$$1 \;\leq\; \#\{(\sigma_c\eta)(N)\cap C_{r_N}\} \;\leq\; 2\,.$$

Exactly one of the three following cases can happen.

- If $\#\{(\sigma_c\eta)(N)\cap C_{r_N}\}=2$, then we set $\eta_{\sigma_c(N+1)}\in C_{r_N+1}$, ie
$$\sigma_c(N+1) \;:=\; \sigma_c^{(r_N+1)}(1)\,,$$
and we see that (4.4) and (4.5) are still satisfied for $N+1$ and all $r\geq 1$.
- Otherwise, one has $\#\{(\sigma_c\eta)(N)\cap C_{r_N}\}=1$ then we want to set $\eta_{\sigma_c(N+1)}$ in $\bigcup_{r=1}^{r_N-1}C_r$ so that (4.4) and (4.5) are still true. From (4.4), one has, for all $r\geq 2$,
$$\#\{(\sigma_c\eta)(N)\cap C_{r-1}\} \;\leq\; 2\#\{(\sigma_c\eta)(N)\cap C_r\}+2\,.$$
Let $s\geq 2$ bee the biggest r, $2\leq r\leq r_N$, if it exists, such that $\#\{(\sigma_c\eta)(N)\cap C_{r-1}\} = 2\#\{(\sigma_c\eta)(N)\cap C_r\}+2$. Then we set $\eta_{\sigma_c(N+1)}\in C_s$, ie
$$\sigma_c(N+1) \;:=\; \sigma_c^{(s)}\left(1+\#\{(\sigma_c\eta)(N)\cap C_s\}\right)\,,$$
and we see that (4.4) and (4.5) are still satisfied for $N+1$ and all $r\geq 1$ (from hypothesis, one also has $\#\{(\sigma_c\eta)(N)\cap C_s\}\leq 2\#\{(\sigma_c\eta)(N)\cap C_{s+1}\}+1$).
- The last remaining case is the following: for all $r=2,\ldots,r_N$, one has $\#\{(\sigma_c\eta)(N)\cap C_{r-1}\}\leq 2\#\{(\sigma_c\eta)(N)\cap C_r\}+1$. Then we just set $\eta_{\sigma_c(N+1)}\in C_1$, ie
$$\sigma_c(N+1) \;:=\; \sigma_c^{(1)}\left(1+\#\{(\sigma_c\eta)(N)\cap C_1\}\right)\,,$$
and we see that (4.4) and (4.5) are still satisfied for $N+1$ and all $r\geq 1$.

Now we can see from (4.4) by induction on $r\leq r_N$ that

$$\frac{\#\{(\sigma_c\eta)(N)\cap C_1\}}{2^{r-1}} \;\geq\; \#\{(\sigma_c\eta)(N)\cap C_r\} \;\geq\; \frac{\#\{(\sigma_c\eta)(N)\cap C_1\}}{2^{r-1}}-1-\frac{1}{2}-\cdots-\frac{1}{2^{r-2}}\,,$$

ie

$$\#\{(\sigma_c\eta)(N)\cap C_r\} \;\sim\; \frac{\#\{(\sigma_c\eta)(N)\cap C_1\}}{2^{r-1}}\,.$$

Since

$$N \;=\; \sum_{r=1}^{r_N}\#\{(\sigma_c\eta)(N)\cap C_r\} \;\sim\; \sum_{r=1}^{r_N}\frac{\#\{(\sigma_c\eta)(N)\cap C_1\}}{2^{r-1}} \;\sim\; 2\#\{(\sigma_c\eta)(N)\cap C_1\}\,,$$

one has, for all $r = 1, \ldots, r_N$,

$$\# \left\{ \left(\sigma_c \eta\right)(N) \cap C_r \right\} \quad \sim \quad \frac{N}{2^r} \,,$$

then (4.3) is satisfied. In particular, since $\# \left\{ \left(\sigma_c \eta\right)(N) \cap C_{r_N} \right\} = 1$ or 2, this yields to

$$r_N \quad \sim \quad \log_{[2]} N \,. \tag{4.6}$$

Now that we have constructed the permutation σ_c, we can prove the theorem.

4.2. **Proof of Theorem 5:** On the following, we will just write the sequence $\eta = \left(\eta_j\right)_{j \geq 1}$ that means the sequence $\sigma_c \eta = \left(\eta_{\sigma_c(j)}\right)_{j \geq 1}$.

Let be $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ and

$$f(z) \quad = \quad \sum_{k,l \geq 0} a_{k,l} z_1^k z_2^l \,,$$

its Taylor expansion. We want to prove that the interpolation formula $E_N(f; \eta)$ converges to f, uniformly on any compact subset. By lemma 7, it is sufficient to prove that $R_N(f; \eta)$ converges to 0, uniformly on any compact subset, where

$$R_N(f;\eta)(z) \quad = \quad \sum_{p=1}^{N} \left(\prod_{j=1, j \neq p}^{N} \frac{z_1 - \eta_j z_2}{\eta_p - \eta_j} \right) \sum_{k+l \geq N} a_{k,l} \eta_p^k \left(\frac{z_2 + \overline{\eta_p} z_1}{1 + |\eta_p|^2} \right)^{k+l-N+1} .$$

In the proof, we will not try to use simplifications between the terms of the sum, else we will directly give an upper bound of each of them. We begin with the following preliminar result.

Lemma 11. *Let $K \subset \mathbb{C}^2$ be a compact subset. One has, for all $N \geq 1$ and all $p = 1, \ldots, N$,*

$$\sup_{z \in K} \left| \prod_{j=1, j \neq p}^{N} \left(z_1 - \eta_j z_2\right) \right| \quad \leq \quad \|z\|_K^{N-1} A^N \,,$$

where $A \geq 1$ is a universal constant and $\|z\|_K = \sup_{z \in K} \sqrt{|z_1|^2 + |z_2|^2}$.

Proof. One has by the Cauchy-Schwarz inequality

$$\sup_{z \in K} \prod_{j=1, j \neq p}^{N} |z_1 - \eta_j z_2| \;\leq\; \|z\|_K^{N-1} \prod_{j=1, j \neq p}^{N} \sqrt{1 + |\eta_j|^2} \;\leq\; \|z\|_K^{N-1} \prod_{j=1}^{N} (1 + |\eta_j|)$$

and

$$\begin{aligned} \prod_{j=1}^{N} (1 + |\eta_j|) \quad &= \quad \prod_{r=1}^{r_N} \left[\prod_{r-1 < |\eta_j| \leq r} (1 + |\eta_j|) \right] \;\leq\; \prod_{r=1}^{r_N} (r+1)^{\# \{\eta(N) \cap C_r\}} \\ &= \quad \exp \left[\sum_{r=1}^{r_N} \# \left\{ \eta(N) \cap C_r \right\} \log(r+1) \right] \;\leq\; \exp \left[\sum_{r=1}^{r_N} 2 \frac{N}{2^r} \log(r+1) \right] \\ &\leq \quad \exp\left(\alpha N\right) \,, \end{aligned}$$

since the series $\sum_{r=1}^{r_N} \dfrac{\log(r+1)}{2^r}$ converges. The lemma follows.

$\surd$

Now we prove another preliminar result.

Lemma 12. *One has, for all $K \subset \mathbb{C}^2$ compact subset, all $R > \|z\|_K$, all $N \geq 1$ and all $p = 1, \ldots, N$,*

$$\sup_{z\in K}\left|\sum_{k+l\geq N} a_{k,l}\eta_p^k \left(\frac{z_2+\overline{\eta_p}z_1}{1+|\eta_p|^2}\right)^{k+l-N+1}\right| \quad \leq \quad \frac{\|f\|_R\|z\|_K}{(1-\|z\|_K/R)^2}(N+1)\left(\frac{1+|\eta_p|}{R}\right)^N ,$$

where $\|f\|_R = \sup_{|z_1|,|z_2|\leq R}|f(z)|$.

Proof. First, we remind by the Cauchy inequalities (3.18) that $|a_{k,l}| \leq \|f\|_R/R^{k+l}$. It follows by the Cauchy-Schwarz inequality that

$$\sup_{z\in K}\sum_{k+l\geq N}|a_{k,l}|\,|\eta_p|^k\left|\frac{z_2+\overline{\eta_p}z_1}{1+|\eta_p|^2}\right|^{k+l-N+1} \quad \leq$$

$$\begin{aligned}
&\leq \quad \sum_{m\geq N}\left(\frac{\|z\|_K\sqrt{1+|\eta_p|^2}}{1+|\eta_p|^2}\right)^{m-N+1}\sum_{k+l=m}\frac{\|f\|_R}{R^{k+l}}|\eta_p|^k \\
&\leq \quad \|f\|_R\left(\frac{\sqrt{1+|\eta_p|^2}}{\|z\|_K}\right)^{N-1}\sum_{m\geq N}\left(\frac{\|z\|_K}{R\sqrt{1+|\eta_p|^2}}\right)^m (m+1)\left(\sqrt{1+|\eta_p|^2}\right)^m \\
&\leq \quad \|f\|_R\left(\frac{1+|\eta_p|}{\|z\|_K}\right)^{N-1}\sum_{m\geq N}(m+1)\left(\frac{\|z\|_K}{R}\right)^m \\
&= \quad \|f\|_R\left(\frac{1+|\eta_p|}{\|z\|_K}\right)^{N-1}\frac{(N+1)(\|z\|_K/R)^N}{(1-\|z\|_K/R)^2},
\end{aligned}$$

and the lemma follows.

$\surd$

Now the next step will be getting a lower bound of each product $\prod_{j=1,j\neq p}^N |\eta_p-\eta_j|$. Since one has

$$\prod_{j=1,j\neq p}^{N}|\eta_p-\eta_j| \;=\; \left(\prod_{|\eta_p-\eta_j|\geq 1/3}|\eta_p-\eta_j|\right)\left(\prod_{|\eta_p-\eta_j|<1/3,j\neq p}|\eta_p-\eta_j|\right), \tag{4.7}$$

then we want to get an estimate of each product. We prove the next result that gives a lower bound of the second one.

Lemma 13. *One has, for all $N \geq 1$ and all $p = 1, \ldots, N$,*

$$\prod_{|\eta_p-\eta_j|<1/3,j\neq p}|\eta_p-\eta_j| \quad \geq \quad 1/B^N ,$$

where $B \geq 1$ is universal.

Proof. Let be $r \geq 1$ such that $\eta_p \in C_r$. We know by construction of σ_c that η_p belongs to an $\varepsilon_{N,r}$-net, where

$$\varepsilon_{N,r} \quad \sim \quad \frac{1}{2^{k_{N,r}}}$$

(the power $k_{N,r}$ depends on N by construction of σ_c and also on r since σ_c is constructed with the partial $\sigma_c^{(r)}$'s). We know from Subsubsection 4.1.1 and (4.2) that

$$\#\left\{\eta(N)\cap C_r\right\} \quad\sim\quad \gamma_{r,k_{N,r}}2^{2k_{N,r}} \;\sim\; 8r\left(1-\frac{1}{2^{k_{N,r}}}\right)2^{2k_{N,r}}\,.$$

On the other hand, by (4.3) from Subsubsection 4.1.2 one also has

$$\#\left\{\eta(N)\cap C_r\right\} \quad\sim\quad \frac{N}{2^r}\,,$$

then

$$\varepsilon_{N,r} \quad\sim\quad \sqrt{\frac{1}{2^{2k_{N,r}}}} \;\sim\; \sqrt{\frac{8r(1-1/2^{k_{N,r}})}{N/2^r}} \;\sim\; \frac{\delta_{N,r}}{\sqrt{N}}\,, \tag{4.8}$$

where

$$\delta_{N,r} \quad\xrightarrow[N\to\infty]{}\quad \sqrt{8r}\,2^{r/2}\,.$$

First, assume that $\operatorname{dist}(\eta_p, bC_r) \geq 1/3$ (bC_r being the topological edge of C_r). It follows that

$$\{\eta_j,\,|\eta_p-\eta_j|<1/3\} \quad\subset\quad \eta(N)\cap C_r\,.$$

Let $s_{N,r}$ the biggest integer $s\geq 1$ such that $s\varepsilon_{N,r}<1/3$, ie

$$s_{N,r}\varepsilon_{N,r} \;<\; 1/3 \;\leq\; (s_{N,r}+1)\,\varepsilon_{N,r}\,. \tag{4.9}$$

It follows that

$$\begin{aligned}\prod_{|\eta_p-\eta_j|<1/3,j\neq p}|\eta_p-\eta_j| \quad&=\quad \prod_{s=1}^{s_{N,r}-1}\;\prod_{s\varepsilon_{N,r}\leq|\eta_p-\eta_j|\leq(s+1)\varepsilon_{N,r}}|\eta_p-\eta_j|\\ &\geq\quad \prod_{s=1}^{s_{N,r}-1}\left(s\varepsilon_{N,r}\right)^{\#\{s\varepsilon_{N,r}\leq|\eta_p-\eta_j|\leq(s+1)\varepsilon_{N,r}\}}\,.\end{aligned} \tag{4.10}$$

By construction of the $\varepsilon_{N,r}$-net, one has

$$\#\{\eta_j,\ \varepsilon_{N,r}\leq|\eta_p-\eta_j|\leq 2\varepsilon_{N,r}\} \quad\preceq\quad 8$$

(ie the upper bound is of order 8). Analogously,

$$\#\{\eta_j,\ 2\varepsilon_{N,r}\leq|\eta_p-\eta_j|\leq 3\varepsilon_{N,r}\} \quad\preceq\quad 16\,,$$

and more generally, one has by induction on $s\geq 1$

$$\#\{\eta_j,\ s\varepsilon_{N,r}\leq|\eta_p-\eta_j|\leq(s+1)\varepsilon_{N,r}\} \quad\preceq\quad 8s\,.$$

It follows by (4.10) that

$$\prod_{|\eta_p-\eta_j|<1/3,j\neq p}|\eta_p-\eta_j| \quad\succeq\quad \exp\left(\sum_{s=1}^{s_{N,r-1}}8s\ln\left(s\varepsilon_{N,r}\right)\right)\,. \tag{4.11}$$

On the other hand, one has by (4.9)

$$\begin{aligned}\sum_{s=1}^{s_{N,r-1}} s\ln(s\varepsilon_{N,r}) \;&\geq\; \sum_{s=1}^{s_{N,r-1}} s\ln\left(\frac{s}{3(s_{N,r}+1)}\right) \;\geq\; \sum_{s=1}^{s_{N,r+1}} s\ln\left(\frac{s}{3(s_{N,r}+1)}\right)\\ &=\; (s_{N,r}+1)^2\,\frac{1}{s_{N,r}+1}\sum_{s=1}^{s_{N,r+1}}\frac{s}{s_{N,r}+1}\ln\left(\frac{s}{3(s_{N,r}+1)}\right).\end{aligned}$$

As $N\to\infty$, by (4.8) $\varepsilon_{N,r}\to 0$ then by (4.9) $s_{N,r}\to\infty$. It follows that the above sum is the Riemann sum of the (Riemann-integrable) function $t\in[0,1]\mapsto t\ln(t/3)$ and whose limit is

$$\frac{1}{s_{N,r}+1}\sum_{s=1}^{s_{N,r+1}}\frac{s}{s_{N,r}+1}\ln\left(\frac{s}{3(s_{N,r}+1)}\right) \;\xrightarrow[N\to\infty]{}\; \int_0^1 t\ln\left(\frac{t}{3}\right)dt \;=\; -\alpha\,,$$

where $\alpha>0$. In particular, the sum is low-bounded then, whatever $s_{N,r}$ is big or not, one gets a lower bound that is uniform on N and $r\geq 1$.

By (4.9) and (4.8), this yields to

$$\sum_{s=1}^{s_{N,r-1}} s\ln(s\varepsilon_{N,r}) \;\succeq\; -\alpha\,(s_{N,r}+1)^2 \;\geq\; -\frac{\alpha}{9\varepsilon_{N,r}^2} \;\succeq\; -\frac{\alpha\,N}{9\delta_{N,r}^2}\,,$$

then (4.11) gives us

$$\prod_{|\eta_p-\eta_j|<1/3,j\neq p}|\eta_p-\eta_j| \;\succeq\; \exp\left(-\frac{8\alpha N}{9\delta_{N,r}^2}\right) \;\geq\; \exp\left(-\frac{\alpha}{9}N\right)\,, \tag{4.12}$$

since by (4.8), one has $\delta_{N,r}^2\geq 8r2^r(1-1/2)\geq 8$.

Now, we assume that $\operatorname{dist}(\eta_p,bC_r)<1/3$, ie $\operatorname{dist}(\eta_p,C_{r-1})<1/3$ or $\operatorname{dist}(\eta_p,C_{r+1})<1/3$. We deal with the first case (the second one is analogous). Between the η_j that are close to η_p some of them can belong to C_{r-1} (then $r\geq 2$). One has

$$\prod_{|\eta_p-\eta_j|<1/3,j\neq p}|\eta_p-\eta_j| = \prod_{|\eta_p-\eta_j|<1/3,\eta_j\in C_r,j\neq p}|\eta_p-\eta_j|\prod_{|\eta_p-\eta_j|<1/3,\eta_j\in C_{r-1}}|\eta_p-\eta_j|\,.$$

By translation of η_p and the $\eta_j\in C_r$ inside of C_r so that $\operatorname{dist}(\eta_p,bC_r)\geq 1/3$, one can complete the partial translated $1/2^{k_{N,r}}$-net and apply (4.12) to get

$$\begin{aligned}\prod_{|\eta_p-\eta_j|<1/3,\eta_j\in C_r,j\neq p}|\eta_p-\eta_j| \;&\geq\; \prod_{|\widetilde{\eta_p}-\widetilde{\eta_j}|<1/3,\widetilde{\eta_j}\in C_r,j\neq p}|\widetilde{\eta_p}-\widetilde{\eta_j}|\\ &\succeq\; \exp\left(-\frac{\alpha}{9}N\right)\,.\end{aligned}$$

Analogously, by translation of η_p inside of C_{r-1} and since (4.12) is uniform on r, one has

$$\prod_{|\eta_p-\eta_j|<1/3,\eta_j\in C_{r-1}}|\eta_p-\eta_j| \;\succeq\; \exp\left(-\frac{\alpha}{9}N\right)\,.$$

This finally yields to

$$\prod_{|\eta_p-\eta_j|<1/3,j\neq p}|\eta_p-\eta_j| \;\succeq\; \exp\left(-\frac{2\alpha}{9}N\right)\,.$$

In addition, the constant does not depend on r, then neither does it on $p=1,\dots,N$, thus the lemma is proved.

$\surd$

Now the next result deals with the first product from (4.7).

Lemma 14. *One has, for all $N \geq 1$ and all $p = 1, \dots, N$,*

$$\prod_{|\eta_p - \eta_j| \geq 1/3} |\eta_p - \eta_j| \quad \geq \quad 1/3^N \,.$$

In addition, if $\eta_p \in C_r$ with $r \geq 3$, then one has a sharper estimate:

$$\prod_{|\eta_p - \eta_j| \geq 1/3} |\eta_p - \eta_j| \quad \geq \quad r^N / D^N \,,$$

where $D \geq 1$ is universal.

Proof. Let fix η_p. The first assertion immediatly follows since $1/3 \leq 1$:

$$\prod_{|\eta_p - \eta_j| \geq 1/3} |\eta_p - \eta_j| \quad \geq \quad \left(\frac{1}{3}\right)^{\#\{\eta_j,\, |\eta_p - \eta_j| \geq 1/3\}} \quad \geq \; 1/3^N \,.$$

Now if we assume that $r \geq 3$, one has

$$\prod_{|\eta_p - \eta_j| \geq 1/3} |\eta_p - \eta_j| \quad = \quad \prod_{|\eta_p - \eta_j| \geq 1/3, \eta_j \in C_{r-1} \cup C_r \cup C_{r+1}} |\eta_p - \eta_j| \; \times \tag{4.13}$$

$$\times \prod_{s=1, s \neq r-1, r, r+1}^{r_N} \; \prod_{\eta_j \in C_s} |\eta_p - \eta_j| \; .$$

On one hand, one still has

$$\prod_{|\eta_p - \eta_j| \geq 1/3, \eta_j \in C_{r-1} \cup C_r \cup C_{r+1}} |\eta_p - \eta_j| \quad \geq \quad \left(\frac{1}{3}\right)^{\#\{\eta(N) \bigcap (C_{r-1} \cup C_r \cup C_{r+1})\}} \tag{4.14}$$

$$\geq \quad 1/3^N \,.$$

On the other hand,

$$\prod_{s=1, s \neq r-1, r, r+1}^{r_N} \; \prod_{\eta_j \in C_s} |\eta_p - \eta_j| \quad = \quad \prod_{s=1}^{r-2} \prod_{\eta_j \in C_s} (|\eta_p - \eta_j|) \prod_{s=r+2}^{r_N} \prod_{\eta_j \in C_s} (|\eta_j - \eta_p|) \,,$$

and for all $s = 1, \dots, r-2$ and all $\eta_j \in C_s$, one has $|\eta_p - \eta_j| \geq r - s - 1$. On the other hand, for all $s = r+2, \dots, r_N$ and all $\eta_j \in C_s$, one just has $|\eta_j - \eta_p| \geq 1$ (the estimate can be made better but it will not be usefull on the following). This yields to

$$\prod_{s=1, s \neq r-1, r, r+1}^{r_N} \; \prod_{\eta_j \in C_s} |\eta_p - \eta_j| \quad \geq \quad \prod_{s=1}^{r-2} (r - s - 1)^{\#\{\eta(N) \cap C_s\}} \prod_{s=r+2}^{r_N} 1^{\#\{\eta(N) \cap C_s\}}$$

$$= \quad \exp\left(\sum_{s=1}^{r-2} \#\{\eta(N) \cap C_s\} \ln(r - s - 1)\right) \,. \tag{4.15}$$

One can get an estimate of the sum as follows:

$$(4.16)\quad \sum_{s=1}^{r-2} \#\{\eta(N)\cap C_s\}\ln(r-s-1) \;=\; (\ln r)\sum_{s=1}^{r-2}\#\{\eta(N)\cap C_s\} \;+ \\ +\sum_{s=1}^{r-2}\#\{\eta(N)\cap C_s\}\ln\left(\frac{r-s-1}{r}\right).$$

By (4.3), the first sum is

$$(4.17)\quad (\ln r)\sum_{s=1}^{r-2}\#\{\eta(N)\cap C_s\} \;\sim\; (\ln r)\sum_{s=1}^{r-2}\frac{N}{2^s} \;=\; \frac{N\ln r}{2}\,\frac{1-1/2^{r-2}}{1-1/2} \\ \sim\; (N\ln r)\left(1-\frac{1}{2^{r-2}}\right).$$

The second one is like

$$\begin{aligned}\sum_{s=1}^{r-2}\frac{N}{2^s}\ln\left(\frac{r-s-1}{r}\right) &= \sum_{1\le s\le r/2-1}\frac{N}{2^s}\ln\left(\frac{r-s-1}{r}\right) + \sum_{r/2-1<s\le r-2}\frac{N}{2^s}\ln\left(\frac{r-s-1}{r}\right)\\ &\ge -(N\ln 2)\sum_{1\le s\le r/2-1}\frac{1}{2^s} - (N\ln r)\sum_{r/2-1<s\le r-2}\frac{1}{2^s}\\ &\ge -N\ln 2 - \frac{N\ln r}{2^{r/2-1}}\sum_{s\ge 0}\frac{1}{2^s},\end{aligned}$$

ie

$$(4.18)\quad \sum_{s=1}^{r-2}\#\{\eta(N)\cap C_s\}\ln\left(\frac{r-s-1}{r}\right) \;\succeq\; -N\ln 2 - \frac{4N\ln r}{2^{r/2}}.$$

Then (4.16), (4.17) and (4.18) yield to

$$\begin{aligned}\exp\left(\sum_{s=1}^{r-2}\#\{\eta(N)\cap C_s\}\ln(r-s-1)\right) &\succeq \exp\left[N\ln r\left(1-\frac{4}{2^r}-\frac{4}{2^r}\right)-N\ln 2\right]\\ &= \frac{r^N}{2^N}\exp\left(-8N\frac{\ln r}{2^r}\right) \;\ge\; \frac{r^N}{(2e^8)^N},\end{aligned}$$

and (4.15) becomes

$$(4.19)\quad \prod_{s=1,s\neq r-1,r,r+1}^{r_N}\ \prod_{\eta_j\in C_s}|\eta_p-\eta_j| \;\succeq\; \frac{r^N}{(2e^8)^N}.$$

Finally, (4.13), (4.14) and (4.19) yield to

$$\prod_{|\eta_p-\eta_j|\ge 1/3}|\eta_p-\eta_j| \;\succeq\; \frac{r^N}{(6e^8)^N},$$

and the lemma is proved.

$\surd$

Lemmas 13 and 14 with (4.7) finally allow us to give a lower bound for the whole product.

Corollary 8. *For all $N \geq 1$ and $p = 1, \ldots, N$, one has*

$$\prod_{j=1, j\neq p}^{N} |\eta_p - \eta_j| \quad \geq \quad 1/(3B)^N .$$

In addition, if $\eta_p \in C_r$ with $r \geq 3$, one has

$$\prod_{j=1, j\neq p}^{N} |\eta_p - \eta_j| \quad \geq \quad r^N/(BD)^N .$$

Now we can give the proof of Theorem 5.

Proof. The function $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ being fixed, for all compact subset $K \subset \mathbb{C}^2$, all $N \geq 1$ and all $R > \|z\|_K$, one has by Lemmas 11 and 12

$$\sup_{z\in K} |R_N(f;\eta)(z)| \quad \leq \tag{4.20}$$

$$\leq \quad \sum_{p=1}^{N} \frac{\sup_{z\in K} \prod_{j=1, j\neq p}^{N} |z_1 - \eta_j z_2|}{\prod_{j=1, j\neq p}^{N} |\eta_p - \eta_j|} \sup_{z\in K} \left| \sum_{k+l\geq N} a_{k,l} \eta_p^k \left(\frac{z_2 + \overline{\eta_p} z_1}{1 + |\eta_p|^2} \right)^{k+l-N+1} \right|$$

$$\leq \quad \|z\|_K^{N-1} A^N \frac{\|f\|_R \|z\|_K}{(1 - \|z\|_K/R)^2} \frac{N+1}{R^N} \sum_{p=1}^{N} \frac{(1 + |\eta_p|)^N}{\prod_{j=1, j\neq p}^{N} |\eta_p - \eta_j|} .$$

Now if $\eta_p \in C_r$ with $r \leq 2$, one has $|\eta_p| \leq r\sqrt{2} \leq 2\sqrt{2}$. It follows by Corollary 8 that

$$\frac{(1 + |\eta_p|)^N}{\prod_{j=1, j\neq p}^{N} |\eta_p - \eta_j|} \quad \leq \quad \frac{(1 + 2\sqrt{2})^N}{1/(3B)^N} \;=\; [3B(1 + \sqrt{2})]^N .$$

If $\eta_p \in C_r$ with $r \geq 3$, then $|\eta_p| \leq r\sqrt{2}$ and by Corollary 8 one can deduce that

$$\frac{(1 + |\eta_p|)^N}{\prod_{j=1, j\neq p}^{N} |\eta_p - \eta_j|} \quad \leq \quad (BD)^N \left(\frac{1 + r\sqrt{2}}{r} \right)^N \quad \leq \quad \left(BD(1 + \sqrt{2}) \right)^N .$$

This allows us to give an estimate for all $p = 1, \ldots, N$, as

$$\frac{(1 + |\eta_p|)^N}{\prod_{j=1, j\neq p}^{N} |\eta_p - \eta_j|} \quad \leq \quad \left(3BD(1 + \sqrt{2}) \right)^N .$$

Finally, (4.20) becomes

$$\sup_{z\in K} |R_N(f;\eta)(z)| \quad \leq \quad \frac{\|f\|_R}{(1 - \|z\|_K/R)^2} \frac{A^N \|z\|_K^N}{R^N} (N+1) \sum_{p=1}^{N} \left(3BD(1 + \sqrt{2}) \right)^N$$

$$= \quad \frac{\|f\|_R}{(1 - \|z\|_K/R)^2}\, N(N+1) \left(\frac{3ABD(1 + \sqrt{2}) \|z\|_K}{R} \right)^N . \tag{4.21}$$

It follows that, for all $R > 3ABD(1 + \sqrt{2})\|z\|_K$ (that is $\geq \|z\|_K$), one has

$$\sup_{z\in K} |R_N(f;\eta)(z)| \quad \xrightarrow[N\to\infty]{} \quad 0 ,$$

and the theorem is proved.

$\surd$

In addition, the estimate (4.21) allows us to get a convergence that is also uniform on any compact subset of holomorphic functions.

Corollary 9. *Let K be a compact subset of $\mathbb{C}^2$ and $\mathcal{K}$ a compact subset of $\mathcal{O}\left(\mathbb{C}^2\right)$ (essentially any family of entire functions that is uniformly bounded on any compact subset of $\mathbb{C}^2$). Then*

$$\sup_{f\in\mathcal{K}}\sup_{z\in K}|R_N(f;\eta)(z)| \xrightarrow[N\to\infty]{} 0\,.$$

It follows that

$$\sup_{f\in\mathcal{K}}\sup_{z\in K}|E_N(f;\eta)(z)-f(z)| \xrightarrow[N\to\infty]{} 0\,.$$

The idea of the construction of σ_c with the ε-nets also gives us an affective way to construct such dense sequences whose associate interpolation formula is convergent. In particular, this justifies Corollary 1 from Introduction.

Remark 4.1. To finish, as said in Introduction, we do not have yet an equivalent criterion for the case of a dense sequence. As a consequence of the direct sense of the proof of Theorem 1 (see [12]) on one hand and by Lemma 10 on the other hand, we know at least the following fact: if $\eta=\left(\eta_j\right)_{j\geq 1}$ makes converge its associate interpolation formula, then for any homographic application h of the same type of (3.21) from Lemma 10, there is $R_h>0$ such that, for all $p,q\geq 0$, one has

$$\left|\Delta_{p,(\theta_p,\ldots,\theta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\theta_{p+1})\right| \quad\leq\quad R_h^{p+q}\,, \tag{4.22}$$

where $\theta_j=h\left(\eta_j\right)$, $j\geq 1$.

This more general condition is always necessary, whatever $\left(\eta_j\right)_{j\geq 1}$ is dense or not (although by Theorem 1 one has a weaker condition that is still sufficient in the case of a non-dense sequence). We think that this new criterion should be also sufficient in the case of a dense family. Moreover, it could be replaced by another one that is weaker: if (4.22) is satisfied with $h_0(\zeta)=\zeta$ and $h_\infty(\zeta)=1/\zeta$, then the interpolation formula $E_N(\cdot;\eta)$ is convergent.

References

[1] C. Alabiso, P. Butera, N-variable rational approximants and method of moments, *J. Mathematical Phys.* **16** (1975), 840–845.

[2] S. Bergman, Über ausgezeichnete Randflächen in der Theorie der Functionen von Zwei komplexen Veränderlichen, *Math. Ann.* **104** (1931), 611-636.

[3] B. Berndtsson, A formula for interpolation and division in $\mathbb{C}^n$, *Math. Ann.* **263** (1983), 399–418.

[4] N. Coleff, M. Herrera, Les courants résiduels associés à une forme méromorphe (French), *Lecture Notes in Mathematics*, **633**, Springer, Berlin (1978).

[5] B.A. Fuks, Special chapters in the theory of analytic functions of several complex variables (In Russian), *Amer. Math. Soc.* (1965).

[6] I. Gel'fand, D. Raikov, G. Shilov, *Commutative normed rings* (In Russian), Chelsea (1964).

[7] G.M. Henkin, A.A. Shananin, Bernstein theorems and Radon transform. Application to the theory of production functions, *Transl. Math. Monogr.* **81** (1990), 189–223.

[8] G.M. Henkin, A.A. Shananin, $\mathbb{C}^n$-capacity and multidimensional moment problem, *Notre Dame Math. Lectures* **12** (1992), 69–85.

[9] M. Herrera, M. Lieberman, Residues and principal values on complex spaces, *Math. Ann.* **194** (1971), 259–294.

[10] L. Hörmander, *An introduction to complex analysis in several variables* (1966).

[11] A. irigoyen, An approximation formula for holomorphic functions by interpolation on the ball (2008), http://arxiv.org/abs/0803.4178

[12] A. Irigoyen, A criterion for the explicit reconstruction of a holomorphic function from its restrictions on lines (2010), http://arxiv.org/abs/1001.2431

[13] B.F. Logan, L.A. Shepp, Optimal reconstruction of a function from its projections, *Duke Math. J.* **42** (1975), 645–659.

[14] *Several Complex Variables I: Introduction to Complex Analysis*, A.G. Vitushkin (ed.), Berlin: Springer (1990).

[15] A.G. Vitushkin, *Theory of the transmission and processing of information*, Pergamon Press (1961).

Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, 08007 Barcelona, Spain

E-mail address: amadeo.irigoyen@ub.edu